# Turbulence and Essential Equivalence of Subspaces

Iian Smythe

Department of Mathematics
Cornell University

GSCL XVI (UIC)
April 26, 2015

# Borel Equivalence Relations

Recall: A Polish space is completely metrizable separable topological space, e.g., $\mathbb{N}$, $\mathbb{R}$, separable Banach spaces, and countable products and $G_\delta$ subsets of these.

An equivalence relation $E$ on $X$ is Borel (analytic) if $E = \{(x,y) \in X^2 : x \mathrel{E} y\}$ is Borel (analytic) as a subset of $X^2$.

If $E$ and $F$ are Borel (analytic) equivalence relations on Polish spaces $X$ and $Y$, respectively, a Borel reduction of $E$ to $F$ is a map $f : X \to Y$ which is Borel and satisfies

$$x \mathrel{E} y \quad \iff \quad f(x) \mathrel{F} f(y)$$

for all $x, y \in X$. In this case, we say that $E$ is Borel reducible to $F$, and write $E \leq_B F$. Intuitively, elements of $X$ can be completely classified up to $E$-equivalence by elements of $Y$ up to $F$-equivalence.

# Borel Equivalence Relations: Examples

## Example

Equality on a Polish space $X$, which we denote by $\Delta(X)$, is a Borel (in fact, closed) equivalence relation.

## Example

$E_0$ is the Borel equivalence relation on $2^\omega$ given by

$$x \; E_0 \; y \quad \Longleftrightarrow \quad \exists n \in \omega \forall m \geq n (x(m) = y(m)).$$

## Example

If $G$ is a Polish group acting continuously on a Polish space $X$, $E_G$ (sometimes $X/G$) is the orbit equivalence relation

$$x \; E_G \; y \quad \Longleftrightarrow \quad \exists g \in G (y = g \cdot x).$$

This is an analytic equivalence relation (it may fail to be Borel).

# $E_0$ and Smooth Classification

A Borel equivalence relation $E$ is smooth if $E \leq_B \Delta(X)$ for some Polish space $X$ (we may assume $X = \mathbb{R}$). That is, $E$-classes can be classified by *real number invariants* in a definable way.

**Proposition (Vitali, 1905, in spirit)**

*$E_0$ is not smooth.*

**Corollary**

*If $E$ is a Borel equivalence relation and $E_0 \leq_B E$, then $E$ is not smooth.*

In fact, it is the canonical obstruction, i.e., the converse of the above proposition is also true (the "Glimm-Effros dichotomy" of Harrington–Kechris–Louveau, 1990).

# Classification by Countable Structures

An analytic equivalence relation $E$ is classifiable by countable structures if it is Borel reducible to the isomorphism relation on the countable models of a theory in a countable language.

This time, with rigor: Conisder the case $\mathcal{L} = \{R\}$, the language with a single binary relation. Each countable (infinite) $\mathcal{L}$-structure $\mathfrak{A}$ can be represented as an $x \in 2^{\omega \times \omega}$ where

$$x(n,m) = 1 \iff R^{\mathfrak{A}}(n,m).$$

$x$ and $y$ represent isomorphic $\mathcal{L}$-structures if and only if there exists $g \in S_\infty$ such that $y = g \cdot x$, where $(g \cdot x)(n,m) = x(g^{-1}(n), g^{-1}(m))$ for all $n, m \in \omega$. Note that this is a continuous action of $S_\infty$ on $2^{\omega \times \omega}$.

A similar construction works for all countable languages.

# Turbulence

Hjorth isolated a *dynamical* condition for Polish group actions which precludes classification by countable structures.

For $G$ a Polish group acting continuously on a Polish space $X$, we say that the action of $G$ is turbulent if

- every orbit is dense;
- every orbit is meager;
- every local orbit is somewhere dense.

**Theorem (Hjorth, 1996)**

*Suppose $G$ acts turbulently on $X$, and $S_\infty$ acts continuously on $Y$. If $f : X \to Y$ is an equivariant Borel map, i.e.,*

$$x \; E_G \; y \quad \Rightarrow \quad f(x) \; E_{S_\infty} \; f(y),$$

*for all $x, y \in X$, then $f$ maps a comeager set $C \subseteq X$ to a single $S_\infty$-orbit.*

# Turbulence (cont'd)

## Corollary

*Let $G$ be a Polish group acting turbulently on a Polish space $X$. Then, $E_G$ is not classifiable by countable structures.*

## Proof.

If $E_G$ was classifiable by countable structures, then there is a Borel reduction $f$ of $E_G$ to $E_{S_\infty}$, for some continuous action of $S_\infty$.
$f$ is equivariant, so by Hjorth's Theorem, there is a comeager set $C \subseteq X$ such that $f$ maps $C$ to a single $S_\infty$-orbit.
But since $f$ is a reduction, $C$ must be contained in a single $G$-orbit. $\square$

## Corollary

*Let $G$ be a Polish group acting turbulently on a Polish space $X$, and $E$ an analytic equivalence relation. If $E_G \leq_B E$, then $E$ is not classifiable by countable structures.*

# Turbulence: Examples

### Example

Let $G$ be a proper Polishable subgroup of $(\mathbb{R}^\omega, +)$, and such that for every $\vec{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n$, there is a $g \in G$ which agrees with $\vec{x}$ on its first $n$ coordinates, e.g., $c_0$ and $\ell^p$ $(1 \leq p < \infty)$. Then the action of $G$ by translation on $\mathbb{R}^\omega$ is turbulent.

### Example

The same holds for $G$ a proper Polishable subgroup of $(\mathbb{T}^\omega, \cdot)$, where $\mathbb{T}$ is the unit circle group.

### Example

If $X$ is a separable infinite dimensional Banach space, and $Y$ a linear subspace of $X$ which is dense and Polishable, then the action of $Y$ on $X$ by translation is turbulent.

# Turbulence: Examples (cont'd)

Consider the equivalence relation $[0,1]^\omega/c_0$, the restriction of the orbit equivalence relation of the translation action of $c_0$ on $\mathbb{R}^\omega$ to $[0,1]^\omega$. Note that this is no longer (a priori) an orbit equivalence relation.

## Proposition

*$[0,1]^\omega/c_0$ is Borel bireducible with $\mathbb{T}^\omega/G_0$, for $G_0 = \{(z_n)_n : \lim_n z_n = 1\}$ the image of $c_0$ under the quotient map $\mathbb{R}^\omega \to \mathbb{T}^\omega$.*

## Proof.

To reduce $[0,1]^\omega/c_0$ to $\mathbb{T}^\omega/G_0$, map $(x_n)_n$ to $(e^{(i\pi/2)x_n})_n$.
For the reverse reduction, reduce $\mathbb{T}^\omega/G_0$ to $([-1,1]^2)^\omega/(c_0 \times c_0)$ by embedding the circle into $[-1,1]^2$.
Contract $[-1,1]$ to $[0,1]$, then alternate coordinates to Borel reduce $([0,1]^2)^\omega/(c_0 \times c_0)$ to $[0,1]^\omega/c_0$. □

# Bounded Operators on a Hilbert Space

Fix a separable infinite dimensional (complex) Hilbert space $H$.

Let $\mathcal{B}(H)$ denote the space of all bounded linear operators on $H$, a C*-algebra with the operations of multiplication (composition), adjoint, and the operator nom.

We will investigate Borel equivalence relations occurring on $\mathcal{B}(H)$.

Caution: $\mathcal{B}(H)$ is not separable in the operator norm. Instead, we consider it with the strong operator topology, in which its Borel structure is standard, though it still fails to be Polish. In fact, there is no Polish topology on $\mathcal{B}(H)$ which makes addition continuous and preserves this Borel structure.

# Essential Equivalence

Recall that an operator $K$ on $H$ is compact if it maps the unit ball of $H$ to a set with compact closure; equivalently $K$ is a norm-limit of finite rank operators. Denote by $\mathcal{K}(H)$ the set of compact operators.

**Fact**

*$\mathcal{K}(H)$ is a proper closed ideal in $\mathcal{B}(H)$. In fact, it is the only one.*

Thus, it is natural to consider equivalence in $\mathcal{B}(H)$ modulo compact operators, or essential equivalence, which we denote by $\equiv_{\text{ess}}$

One can check that this equivalence relation is Borel in the standard Borel structure on $\mathcal{B}(H)$.

# Essential Equivalence (cont'd)

## Proposition

$[0,1]^\omega / c_0 \leq_B \equiv_{ess}$. *Thus,* $\equiv_{ess}$ *is not classifiable by countable structures.*

## Proof.

Fix an orthonormal basis $(e_n)_{n\in\omega}$ for $H$. Consider the map $[0,1]^\omega \to \mathcal{B}(H)$ given by $\alpha = (\alpha_n)_n \mapsto T_\alpha$, where for $v \in H$

$$T_\alpha v = \sum_{n=0}^{\infty} \alpha_n \langle v, e_n \rangle e_n.$$

Such operators are diagonal with eigenvalues $\alpha_n$, and it is a standard fact that a diagonal operator is compact if and only if the sequence of eigenvalues converges to $0$. Applying this to $T_\alpha - T_\beta$ for $\alpha, \beta \in [0,1]^\omega$ shows that this map is a reduction. It is easily seen that the map is continuous, hence Borel. □

# Motivating Theorems

Motivating our study of equivalence relations on operators:

**Theorem (Weyl–von Neumann, 1930's)**

*If $S$ and $T$ are bounded self-adjoint operators on $H$, then $S$ and $T$ are unitarily equivalent modulo compact if and only if $S$ and $T$ have the same essential spectrum.*

**Theorem (Ando–Matsuzawa, 2014)**

*The Weyl–von Neumann correspondence is a Borel reduction from unitary equivalence modulo compact of self-adjoint operators to equality on closed subsets $\mathbb{R}$.*

**Theorem (Kechris–Sofranidis, 2001)**

*The conjugation action of the unitary group $U(H)$ on itself, and on self-adjoint operators of norm $1$, is (generically) turbulent.*

# Subspaces and Projections

A projection $P \in \mathcal{B}(H)$ is an operator satisfying $P = P^2 = P^*$. There is a bijective correspondence between projections and closed subspaces of $H$ given by $P \longleftrightarrow \operatorname{ran}(P)$. Denote by $\mathcal{P}(H)$ the set of projections.

We will consider the restriction of $\equiv_{\text{ess}}$ to $\mathcal{P}(H)$, yielding a notion of essential equivalence of subspaces.

**Fact**

*$\mathcal{P}(H)$ is a Polish space in the strong operator topology, and $\equiv_{\text{ess}}$ is a Borel equivalence relation on $\mathcal{P}(H)$.*

# Essential Equivalence of Projections

## Proposition

*$E_0$ is Borel reducible to $\equiv_{ess}$ on $\mathcal{P}(H)$.*

## Proof.

Fix an orthonormal basis $(e_n)_{n\in\omega}$. Define the map $2^\omega \to \mathcal{P}(H) : x \mapsto P_x$ where $P_x$ is the projection onto $\overline{\text{span}}\{e_n : n \in x\}$. For $v \in H$,

$$P_x v = \sum_{n=0}^{\infty} x_n \langle v, e_n \rangle e_n.$$

For $x, y \in 2^\omega$, $(P_x - P_y)v = \sum_{n=0}^{\infty} (x_n - y_n)\langle v, e_n\rangle e_n$, for all $v \in H$. Again, this diagonal operator is compact if and only if $x_n - y_n \to 0$, but since $x_n - y_n \in \{-1, 0, 1\}$ for all $n$, this occurs if and only if $x_n = y_n$ for all but finitely many $n$. □

# A Twist for Non-Classifiability

But in fact, we can show much more:

**Theorem**

*$[0,1]^\omega/c_0$ is Borel reducible to $\equiv_{ess}$ on $\mathcal{P}(H)$. Consequently, the latter is not classifiable by countable structures.*

# A Twist for Non-Classifiability (cont'd)

The reduction of $[0,1]^\omega/c_0$ to $\equiv_{\mathrm{ess}}$ on $\mathcal{P}(H)$ is given by the map $[0,1]^\omega \to \mathcal{P}(H) : \alpha = (\alpha_n)_n \to P_\alpha$, where $P_\alpha$ is the projection onto $\overline{\mathrm{span}}\{e_{2n} + \alpha_n e_{2n+1} : n \in \omega\} = \overline{\mathrm{span}}\{\frac{1}{\sqrt{1+\alpha_n^2}}(e_{2n} + \alpha_n e_{2n+1}) : n \in \omega\}$.

The flavor of the proof: we establish a decomposition for $P_\alpha - P_\beta$:

$$P_\alpha - P_\beta = T_0 + S_0 T_1 + S_1 T_2 + T_3,$$

where, for $v \in H$

$$T_0 v = \sum_{n=0}^{\infty} \left[ \frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2} \right] \langle v, e_{2n} \rangle e_{2n}, \qquad T_2 v = \sum_{n=0}^{\infty} \left[ \frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2} \right] \langle v, e_{2n} \rangle e_{2n},$$

$$T_1 v = \sum_{n=0}^{\infty} \left[ \frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2} \right] \langle v, e_{2n+1} \rangle e_{2n+1}, \quad T_3 v = \sum_{n=0}^{\infty} \left[ \frac{\alpha_n^2}{1+\alpha_n^2} - \frac{\beta_n^2}{1+\beta_n^2} \right] \langle v, e_{2n+1} \rangle e_{2n+1},$$

$$S_0 v = \sum_{n=0}^{\infty} \langle v, e_{2n+1} \rangle e_{2n}, \qquad S_1 v = \sum_{n=0}^{\infty} \langle v, e_{2n} \rangle e_{2n+1}.$$

# A Twist for Non-Classifiability (cont'd)

The upshot of the decomposition

$$P_\alpha - P_\beta = T_0 + S_0 T_1 + S_1 T_2 + T_3$$

is that $T_0$, $T_1$, $T_2$, and $T_3$ are diagonal operators whose eigenvalues go to $0$ when $\alpha_n - \beta_n \to 0$. Since the compact operators form an ideal, this implies that $P_\alpha - P_\beta$ is compact.

To obtain that $P_\alpha - P_\beta$ compact implies $\alpha_n - \beta_n \to 0$, one uses that if the former holds, then $(P_\alpha - P_\beta)e_n \to 0$ in norm, and that

$$(P_\alpha - P_\beta)e_{2n} = \left[\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2}\right] e_{2n} + \left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2}\right] e_{2n+1},$$

$$(P_\alpha - P_\beta)e_{2n+1} = \left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2}\right] e_{2n} + \left[\frac{\alpha_n^2}{1+\alpha_n^2} - \frac{\beta_n^2}{1+\beta_n^2}\right] e_{2n+1}.$$

From here, orthogonality of $e_{2n}$ and $e_{2n+1}$ and a series of inequalities yield that $\alpha_n - \beta_n$ must converge to $0$ if the displayed sequences do.

# Related Results

$E_1$ is the Borel equivalence relation of equality modulo finitely many coordinates on $[0,1]^\omega$.

### Theorem

*$E_1$ is Borel reducible to equivalence modulo finite dimensions in $\mathcal{P}(H)$. Consequently, the latter is not Borel reducible to the orbit equivalence relation of any Polish group action.*

### Theorem

*For $1 \leq p < \infty$,*

1. *$[0,1]^\omega/\ell^p$ is Borel bireducible to the orbit equivalence relation of the turbulent action of $G_p = \{(z_n)_n \in \mathbb{T}^\omega : \sum_{n=0}^{\infty} |\mathrm{Arg}(z_n)|^p < \infty\}$ by translation on $\mathbb{T}^\omega$.*
2. *$[0,1]^\omega/\ell^p$ is Borel reducible to equivalence modulo the Schatten $p$-ideal in $\mathcal{B}(H)$ (or $\mathcal{K}(H)$).*

Smythe, I. B. *Borel equivalence relations in the space of bounded operators*. arXiv:1407.5325 [math.LO]. 2014. (submitted)

Thanks for listening!