MATH 1340 — Mathematics & Politics

Lecture 1 — June 22, 2015
Course Information

- Instructor: lian Smythe
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  112 Malott Hall

- Office hours: M 1:00-3:00pm, Th 2:00-3:00pm in 112 Malott Hall (or by appointment)

- TA/Grader: Sergio Da Silva
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- Office hours: W 10:00am-11:00am in 218 Malott Hall
Course Information (cont’d)

- Lectures: MTWThF 11:30am-12:45pm 224 Malott Hall


- Website (for homework, slides, etc): www.math.cornell.edu/~ismythe/S15_1340.html
Course Outline

- Voting and Social Choice (Part I of R&U)
  - How do we choose a winning candidate in an election?
  - Voting methods (“social choice functions”)
  - Two candidates?
  - Multiple candidates?
  - What characteristics do we want from a voting method?
  - Is it possible to have all of these characteristics? (Arrow’s Theorem)
Course Outline

• Game Theory (Part III of R&U)
  • How do you maximize your payoff in a competitive game knowing that your opponent is trying to do the same?
  • Zero-sum and matrix games
  • Probabilistic ("mixed") strategies
  • Nash equilibria
  • The Prisoner’s Dilemma

• Other topics (Parts II & IV of R&U, others)
  • Apportionment, the Electoral College, gerrymandering
Evaluation

- Homework (40%) will be assigned each Tuesday (due on Friday, in class), and each Friday (due the following Tuesday, in class). The lowest two homework scores will be dropped. See syllabus for more information.

- Test 1 (25%) on Voting and Social Choice (tentatively scheduled for July 7, in class).

- Test 2 (25%) on Game Theory (scheduled for August 3, at 1:30pm, in 224 Malott Hall).

- Class participation (10%). See syllabus for information.
Voting and Social Choice
Electing a class president

• Suppose that we must elect a class president from a slate of two candidates.

• We need to choose a voting method or social choice function, that is, a method which selects a winning candidate (or candidates) given the votes of the class.

• For simplicity, we assume that everyone in the class must casts a vote by writing the name of one of the candidates on their ballot.

  • No blank ballots, write-ins, abstentions, etc.
Electing a class president (cont’d)

• What method should we use to pick the winner(s)? Why?

• Things that came up in class:
  • pick the candidate with the most votes; seems most likely to make the most people happy.
  • What if they get the same number of votes? Drop one of the votes? Toss a coin? Have a tie?
  • What if one of the candidates decided that he wins, no matter what?
  • What if we weighted different students’ votes differently?
  • We want a method that seems fair.
Voting and Social Choice

- What this section is not about:
  - who the candidates are, their policies, speeches, debates, etc
  - who the voters are, registration, suffrage, voter suppression, etc
  - how votes are cast, voting machines, etc

- What it is about:
  - once the votes are cast, how do we determine the winner(s)?
Aside on definitions

• Throughout this course, we will need to precise about our definitions. Thus, terms will be in **bold** when they are being precisely defined on these slides.

• When there is a (rare) conflict between a definition in the text and a definition in the slides, we will default to using the definition in the slides.

• Often, the definition of a term may clash with its common usage. Again, we will default to using the definition in the slides.
Ballots and profiles

- We begin with the case of two candidates, A and B.
  - One candidate elections are… not that interesting.
  - Candidates need not be people; they can be choices, yes or no, etc.

- We assume that all voters (the electorate), which we may number as 1, 2, 3,…, submit a ballot with either A or B on it (not both). The collection of all ballots is called a profile.

- We do not allow write-ins, abstentions, blank ballots, etc.
Social choice functions — Two candidates

• A function is a rule that assigns to every possible input from one set (called the domain) a single output in another set (called the codomain).

  • Functions cannot be indifferent: they must always output something on a given input from the domain.

  • Functions cannot change their minds: if a function outputs “apple” on input “cow”, it must always output “apple” on input “cow”.

• A social choice function (or voting method) is a function with domain the set of all possible profiles from a fixed electorate, and codomain the set consisting of “A wins”, “B wins”, and “tie”. 
(Non-)examples

• Suppose we were to pick a candidate by flipping a coin, say heads, A wins, and tails, B wins. Is this a social choice function, as we have defined them?

• Nope! This method could pick different winners for the same profile. These sorts of “probabilistic methods” are not allowed.

• How about the method that picks the candidate, A or B, with the most votes?

• If there is an even number of voters, this isn’t one either, because it doesn’t output anything if the candidates receive the same number of votes.
Simple-majority method

- The **simple-majority method** is the social choice function that, given a profile as input, outputs the candidate with the *most* number of votes, or “tie” if both candidates get the *same* number of votes.

### Input

<table>
<thead>
<tr>
<th>A</th>
<th>A</th>
<th>B</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>B</td>
<td>A</td>
<td>A</td>
</tr>
</tbody>
</table>

### Output

- **A wins**
- **Tie**
Tabulated profiles

• Observe that the simple majority method depends only on the *number* of votes cast for each of the candidates, not who cast them.

• So, we can simplify the input to a **tabulated profile**, listing only the candidates and the number of votes.

<table>
<thead>
<tr>
<th>Profile</th>
<th>Tabulated profile</th>
</tr>
</thead>
<tbody>
<tr>
<td>A  A  B  A  B</td>
<td>A</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

• Can you think of situations where *who* casts the ballots should matter?
Super-majority methods

• Sometimes a bare majority is not regarded as sufficient; can you think of examples?

• Let $p$ be a number with $1/2 < p \leq 1$. The super-majority method with parameter $p$ is the social choice function that, given a profile as input, outputs the candidate whose fraction of the total vote is at least $p$, and “tie” otherwise.

  • If there are $t$ voters, then the super-majority method selects as the winner the candidate with at least $pt$ votes, and a tie otherwise.

  • The super-majority method also depends only on the tabulated profile.

• When $p=1$, this method requires unanimity to produce a unique winner.
Quota methods

• A related method to the super-majority is the quota method with quota $q$ (some whole number), which choses as the winner(s) the candidate(s) with at least $q$ many votes.

• A super-majority method with parameter $p$ is “functionally equivalent” (next time) to a quota method with quota $q = \text{the smallest integer greater than } pt$.

• What happens if the quota is less than half of the size of the electorate in a two candidate election?
A silly method?

- The **simple-minority method** is the social choice function that, given a profile as input, outputs the candidate with the *least* number of votes, or “tie” if both candidates get the *same* number of votes.

- This may seem like a useless method, but never-the-less, it remains a valid social choice function.
Status quo methods

• In some cases, such as when voting Y or N on a piece of legislation, allowing ties makes no sense.

• We may designate one of the “candidates” as the status quo, and the other as the challenger, and use a social choice function (e.g., simple-majority, super-majority) to first determine if there is a unique winner, and if so, select that winner. If not, we select the status quo. This is a status quo method.
Status quo methods (cont’d)

• For example: When a bill is being considered in the US Senate, it requires a super-majority of 3/5 of all senators to end a filibuster.

• Considering Y (end the filibuster) and N (continue the filibuster) as our candidates, this describes a super-majority with status quo method, with parameter $p = \frac{3}{5}$, and status quo candidate N.
Status quo methods (cont’d)

- Since there are 100 senators, that means \(100 \times \frac{3}{5} = 60\) votes of Y are required to end the filibuster, otherwise, the filibuster (being the status quo) continues.

<table>
<thead>
<tr>
<th>Y</th>
<th>N</th>
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<tbody>
<tr>
<td>57</td>
<td>43</td>
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</table>

Filibuster goes on

<table>
<thead>
<tr>
<th>Y</th>
<th>N</th>
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<tbody>
<tr>
<td>60</td>
<td>40</td>
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Filibuster ends
Weighted voting method

- The **weighted voting method** is as follows: Suppose that there are \( n \) voters, labeled 1, 2,\ldots, \( n \) and voter \( i \) is given a positive number \( w_i \) of votes (called the **weight** of voter \( i \)). Let \( t = w_1 + \cdots + w_n \) be the total number of votes. This method selects as winner(s) the candidate(s) who receive more than half (i.e., \( t/2 \)) of the votes. If no such candidate exists, the result is a tie.
Weighted voting method (cont’d)

• For example: Suppose that there are five voters with weights 5, 5, 3, 2, 2 respectively. Consider the following profiles (with voters listed in the above order). Note that we cannot use tabulated profiles in this method.

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<table>
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</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>B</td>
<td>B</td>
<td>B</td>
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</table>

Who wins?  A

<p>| | | | | |</p>
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<tr>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>B</td>
</tr>
</tbody>
</table>

Who wins?  B
Functional equivalence

• Sometimes, two social choice functions with different descriptions may actually be, *for all intents and purposes*, the same.

• We say that two social choice functions are *functionally equivalent* if whenever they are given the same input profiles, they produce the same result.

  • A simple example is weighted voting with every voter getting weight $= 1$. This is functionally equivalent to the simple majority method.

• **Caution:** This notion is not in the text (explicitly), but we will use it repeatedly.
Functional equivalence (cont’d)

• For example: consider the weighted voting method with five voters having weights $12, 10, 9, 9, 8$. Note that $t=48$, so more than $t/2=24$ votes are required to win.

• We claim that this method is functionally equivalent to the simple-majority method. Why?

• From class: In the simple majority method, the winner is whoever gets 3 or more votes (no ties possible, since there are 5 voters).

• In this example, every combination of 3 votes has weight at least 26, since the smallest such combination is $9+9+8=26 > 24$. Thus, whoever receives any combination of 3 votes is the winner, and this is the only possible way to win, since combinations of 2 or fewer votes is at most $12+10=22 < 24$. Thus, this method is functionally equivalent to the simple majority method.
Other social choice functions

• We can combine the methods we’ve seen to create new social choice functions.

• Example 1.8 in R&U: Suppose that a small business has six partners, who vote on decisions with weights 10, 5, 5, 3, 2, and 1, respectively, and a decision requires a $p = \frac{3}{5}$ super-majority to pass. Otherwise, the decision is rejected. Suppose that the partners are voting on a decision A, with alternative B being to reject the amendment. Does A pass with the following profile?

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>A</th>
<th>A</th>
<th>A</th>
<th>B</th>
<th>A</th>
</tr>
</thead>
</table>
Other social choice functions (cont’d)

Continuing the example: The total number of weighted votes is $t = 10 + 5 + 5 + 3 + 2 + 1 = 26$.

At least $(3/5)\times(26) = 15.6$ weighted votes are required to pass, that is, 16 weighted votes are required. (We can’t cut voters in half, and 15 weighted votes would not be enough.)

Candidate A receives $5 + 5 + 3 + 1 = 14$ weighted votes, which is less than 16, and thus B (rejecting A), being the status quo, prevails.
Other social choice functions (cont’d)

- The **dictatorship method** is the social choice function in which a particular *voter* is chosen to be the *dictator*, and the winner is whichever candidate the dictator choses.

- The **monarchy method** is the social choice function in which a particular *candidate* is chosen to be the *monarch*, and is the winner regardless of the votes.

- The **parity method** is the social choice function which selects as the winner(s) the candidate(s) with an even number of votes, and a tie if no candidate does.

- **Caution:** This is far from an exhaustive list!
• Recommended reading: the syllabus, Sections 1.1-1.2 of R&U