MATH 1340 — Mathematics & Politics

Lecture 10 — July 6, 2015
Arrow’s Theorem

**Arrow’s Impossibility Theorem (1950):** If a social choice function with at least three candidates satisfies Pareto and independence, then it is (functionally equivalent to) a dictatorship.

**Corollary:** It is impossible for a social choice function with at least three candidates to satisfy Pareto, independence and anonymity.

- For the proof, see section 5.4 of R&U or your notes from class.
A different way?

- One response to Arrow’s theorem is to settle: Plurality (or Copeland, or…?) is good enough, and independence is too much to hope for.

- Another response is to start over: Why do we need to use preference ballots?

- If we replace our preference ballots with something different, can we do better?
Approval ballots (a brief tour)

- An **approval ballot** is a ballot in which a voter express their “yes” (Y) or “no” (N) approval of each candidate.

- An **approval profile** is the list of all approval ballots from all of the voters in a given electorate.

- **Tabulated approval profiles** can also be defined.

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Approval ballot of voter 3

Approval profile
Approval social choice functions

- An **approval social choice function** is a function having domain all approval profiles of a fixed electorate, and codomain all subsets of the slate of candidates.
  - Unlike with preference ballots, we allow the possibility of outputting the empty set; i.e., choosing no winners. This may be reasonable when all of the voters disapprove of all of the candidates.

- The **plurality (approval) method** is the approval social choice function that selects as the winner(s) whichever candidate(s) get the most “yes” votes.
  - This method always outputs some winner, even if every vote is N (then, all of the candidates tie).
  - It is used to elect the Secretary General of the United Nations, as well as the presidents of several scientific societies including the Mathematical Association of America and the American Mathematical Society.
We say that a candidate A is **socially preferred** to B (in a given profile) if the number of approval votes for A is greater than the number of approval votes for B.

In the following way, approval ballots avoid the Condorcet paradox:

**Proposition:** If in a given approval profile, A is socially preferred to B, and B is socially preferred to C, then A is socially preferred to C.

Why? “Greater than” is itself transitive.
Criteria for approval methods

Here are some desirable criteria for approval social choice functions:

• An approval social choice function is **anonymous** if the outcome is unchanged whenever two voters exchange approval ballots.

• An approval social choice function is **neutral** if whenever A is a winner in one profile, and all voters who approved of A but not some other candidate B change to approve of B but not A, and and all voters who approved of B but not A change to approve of A but not B (everything else remains unchanged), then B becomes a winner.

• An approval social choice function is **monotone** if whenever a winner gets more “yes” votes, they remain a winner.

• An approval social choice function is **nearly decisive** if whenever it names two winners, they must have exactly the same number of “yes” votes.
Criteria for approval methods (cont’d)

• An approval social choice function is **independent** if whenever we are given two profiles in which every voter has the same opinion of candidates A and B, and A wins and B loses in the first profile, then B must not win in the second profile.

• This criterion is substantially less “mysterious” in the approval ballot setting.

• One can check (it’s not too hard):

  **Proposition:** The plurality method is anonymous, neutral, monotone, nearly decisive, and independent.
Criteria for approval methods (cont’d)

• In fact, more is true:

**Theorem:** The only approval social choice function that is anonymous, neutral, monotone, independent, nearly decisive, and that always selects at least one winner is the plurality method.

• See p. 102 of R&U for the details.
• Recommended reading: Section 5.4 in R&U

• Optional reading: Chapter 6 (specifically 6.3 on Approval Ballots) and the “Notes on Part I” in R&U

• Problem set #4 is due tomorrow, in class.

• Test 1 on Voting and Social Choice Theory is on Wednesday, in class. Please review the end of the slides from Lecture 9 for details. (Note: Approval voting is not part of the test material.)