MATH 1340 — Mathematics & Politics

Lecture 14 — July 10, 2015

Apportionment methods

- Recall that an **apportionment method** is a function which takes as input the values $h, n, p_1, p_2, \ldots, p_n$, where h and n are positive integers, p_k's are positive numbers, and whose output is a sequence of *nonnegative integers* $a_1, a_2, \ldots, a_n$ such that $h = a_1 + a_2 + \ldots + a_n$.

- The interpretation is that given h objects, and n states with populations $p_1, p_2, \ldots, p_n$, the method distributes a_k objects to the k^{th} state.

- In the case of US Congressional apportionment, $h = 435$ (number of seats in the House), $n = 50$ (number of states), and we may order the states alphabetically so that p_k is the population of the k^{th} state in alphabetical order

Criteria and paradoxes

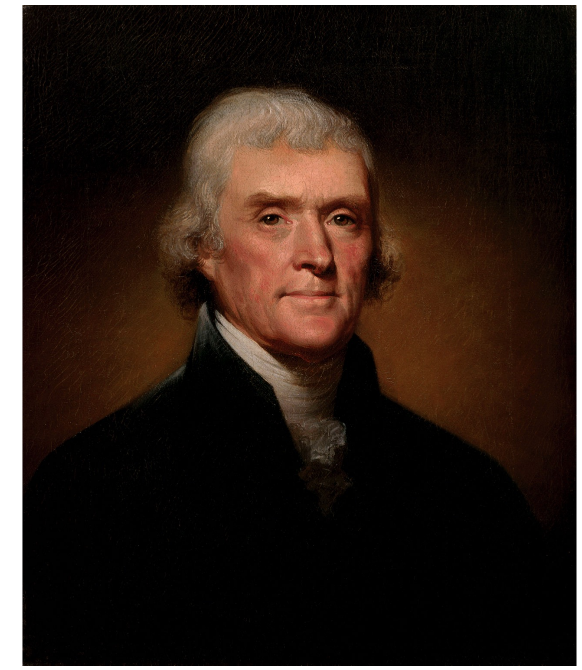
- An **Alabama paradox** is when a state loses a seat when h is increased and all other parameters (number of states, populations) are fixed. An apportionment method that avoids this is called **house monotone**.
- A **population paradox** is when one state gains (or remains the same) in population while another loses (or remains the same), yet it is the first state that loses a seat, while the other gains a seat. Methods that avoid this are said to be **population monotone**.
- A population paradox which results from the addition of a new state is a **new states**, or **Oklahoma**, **paradox**.
- We have seen that Hamilton's method is susceptible to each of these paradoxes, and is thus neither house nor population monotone.

Quotas and divisors

- Given $h, n, p_1, p_2, \ldots, p_n$:
- The quantity $q_k = h(p_k/p) = p_k/(p/h)$ is the **standard quota** for the k^{th} state.
- The **lower quota** is the result of rounding down q_k.
- The **upper quota** is the result of rounding up q_k.
- The quantity $s = p/h$ is the **standard divisor**. This is an ideal amount of the population entitle to each object.

Jefferson's method

- The method below, proposed by Thomas Jefferson (founding father, 3rd President) in response to Hamilton's method, was used for apportionment in the early decades of the US.



Thomas Jefferson
1743-1826

- **Jefferson's method** is as follows:
  - Choose a number d, called the **modified divisor**, which represents a *desired* approximate size for congressional districts.
  - Compute the **modified quotas** p_k/d for each state, and round these numbers *down* to obtain a_k.
  - If $a_1 + a_2 + \ldots + a_n = h$, then we have the apportionment.
  - Otherwise, change d and try again.

Jefferson's method (cont'd)

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- In practice it is not difficult to determine an appropriate value for d:
  - If a chosen d is too large, then the sum $a_1+a_2+\ldots+a_n$ will be smaller than h.
  - If a chosen d is too small, then the sum $a_1+a_2+\ldots+a_n$ will be larger than h.
  - For example, setting $d = s$, the standard divisor, will make a_k equal to the lower quota, so $a_1+a_2+\ldots+a_n$ will be smaller than h, meaning that this divisor is too large.
  - We hone in on a range of values for d that will make $a_1+a_2+\ldots+a_n = h$.

Jefferson's method (cont'd)

- For example: Suppose that $n = 3$, $h = 10$, $p_1 = 1{,}500{,}000$, $p_2 = 3{,}200{,}000$ and $p_3 = 5{,}300{,}000$. Compute the Jefferson apportionment.

k	p_k	standard quota	Hamliton apportion-ment	Jefferson apportion-ment
1	1,500,000	1.5	2	1
2	3,200,000	3.2	3	3
3	5,300,000	5.3	5	6

Jefferson's method (cont'd)

- There may be many different modified divisors d which yield a Jefferson apportionment. However, these apportionments will always be the same!

<u>Proposition:</u> *For h, n, $p_1, p_2, \ldots, p_n$ given, if d and d' are two different divisors yielding Jefferson apportionments $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ respectively, then $a_k = b_k$ for all k.*

- So, as soon as we find a divisor d for which $a_1 + a_2 + \ldots + a_n = h$ we have found *the* (one and only) Jefferson apportionment.

- R&U discuss a more systematic way to determine a good value for d in Section 8.2. We will skip this.

Jefferson's method (cont'd)

- Consider an apportionment with $n = 4$, $h = 10$, $p_1 = 1{,}500{,}000$, $p_2 = 1{,}400{,}000$, $p_3 = 1{,}300{,}000$ and $p_4 = 5{,}800{,}000$. The standard divisor is $s = 1{,}000{,}000$. Use a modified divisor of $d = 800{,}000$:

k	p_k	standard quota	lower quota	upper quota	Hamliton apportion-ment	modified quota	Jefferson apportion-ment
1	1,500,000	1.5	1	2	2	1.88	1
2	1,400,000	1.4	1	2	1	1.75	1
3	1,300,000	1.3	1	2	1	1.62	1
4	5,800,000	5.8	5	6	6	7.25	7

- The Jefferson apportionment to state 4 is greater than its upper quota!

Jefferson's method (cont'd)

- Recall that an apportionment method satisfies the **quota rule** if it always apportions to each state either its upper or lower quota.
- The previous example shows that Jefferson's method fails the quota rule.
- One can also have a **lower quota rule**, no state is assigned less than its lower quota, and an **upper quota rule**, no state is assigned more than its upper quota.

Proposition: *Jefferson's method satisfies the lower quota rule, but not the upper quota rule.*

Jefferson’s method (cont’d)

- However, Jefferson’s method avoids the paradoxes we’ve seen with Hamilton’s method.

<u>Proposition:</u> *Jefferson’s method is House monotone.*

<u>Proposition:</u> *Jefferson’s method is population monotone.*

- We add that, generally speaking, Jefferson’s method tends to favor large states.

Other divisor methods

- Jefferson's method is an example of a **divisor method**, that is, a method in which state populations are divided by a modified divisor to obtain quotas, which are then rounded (*in some way*), to obtain a possible apportionment.

- **Adams' method** (named for John Quincy Adams, 6th President) is the divisor method as follows:
  - Choose a modified divisor d.
  - Compute the modified quotas p_k/d for each state, and round these numbers *up* to obtain a_k.
  - If $a_1+a_2+\ldots+a_n = h$, then we have the desired apportionment.
  - Otherwise, change d and try again.

- Note that Adam's method always apportions at least 1 seat to each state, unlike Jefferson's method. (We require $h \geq n$ for Adams'.)

- Adams' method tends to favor smaller states.

Other divisor methods (cont'd)

- **Webster's method** (named for Daniel Webster, a senator) is the divisor method as follows:
  - Choose a modified divisor d.
  - Compute the modified quotas p_k/d for each state, and round these numbers (in the usual way) to obtain a_k.
  - If $a_1 + a_2 + \ldots + a_n = h$, then we have the desired apportionment.
  - Otherwise, change d and try again.

- Webster's method was used in the late 19th and early 20th century for apportionment (often when it agreed with Hamilton's method)

Other divisor methods (cont'd)

- For example: Suppose that $n = 3$, $h = 10$, $p_1 = 1{,}500{,}000$, $p_2 = 3{,}200{,}000$ and $p_3 = 5{,}300{,}000$. Compute the Adams and Webster apportionments.

k	p_k	standard quota	Hamliton apportion-ment	Jefferson apportion-ment	Adams apportion-ment	Webster apportion-ment
1	1,500,000	1.5	2	1	2	2
2	3,200,000	3.2	3	3	3	3
3	5,300,000	5.3	5	6	5	5

Geometric rounding

- We mention one last divisor method, as it is the method which has been used for congressional apportionment (and is mandated by law) since 1941.

- **Geometric rounding** is the rounding method which, given a number x between integers n and $n+1$, rounds x to n if $x < \sqrt{(n(n+1))}$, and otherwise rounds x to $n+1$.

- For example: If x is between 1 and 2, but less than $\sqrt{2}$, then it is rounded down in this method.
  If x is between 2 and 3, but less than $\sqrt{6}$, then it is rounded down in this method, and so on.

Hill's method

- **Hill's method** (or the **Huntington—Hill method**, named for mathematician Edward Huntington and statistician Joseph Hill) is the divisor method as follows:
  - Choose a modified divisor d.
  - Compute the modified quotas p_k/d for each state, and round these numbers *geometrically* to obtain a_k.
  - If $a_1+a_2+\ldots+a_n = h$, then we have the desired apportionment.
  - Otherwise, change d and try again.

Joseph A. Hill
1860-1938

- Hill's method is often argued for on the basis that it is the *unique* apportionment method which guarantees that no additional transfer of a seat from one state to another will reduce the ratio between degrees of representation (persons per representative) in any two states.

- However, different arguments can also be given for Webster's method.

Divisor methods

- One can show that all of these divisor methods avoid the paradoxes from last class. That is,

<u>Proposition:</u> *Divisor method are house and population monotone.*

- But, they may violate the quota rules. In fact:
  - Jefferson satisfies lower quota, violates upper quota.
  - Adams satisfies upper quota, violates lower quota.
  - Hill and Webster violate both quota rules.

A perfect method?

- We have seen that Hamilton's method satisfies the quota rule, but violates population (and house) monotonicity.
- Meanwhile, divisor methods satisfy population (and house) monotonicity, but may violate the quota rules.
- Can we find a reasonable method which satisfies all of these properties?
- "Reasonable" here means **neutral**, that is, if state i and state j exchange populations (and everything else remains the same), then state i and state j must exchange apportioned amounts.

Balinski—Young Theorem

- Much like in the case of voting methods, our hopes are dashed by a result of the mathematicians Michel Balinski and Peyton Young:

Theorem (Balinski—Young, 1982): It is impossible for a neutral apportionment method to satisfy both the quota rule and population monotonicity.

- The proof, which is not too difficult, is on p. 179 of R&U.

- Recommended reading: Sections 8.1, 8.3-8.5 in R&U
- Optional reading: Anything else in Part II of R&U, particularly the proof of the Balinski—Young theorem in Section 9.6, and Ch. 12 on the History of Apportionment in the US.
- <u>Problem set 5</u> has been posted on the course website, and is due on Tuesday, July 14, in class.