MATH 1340 — Mathematics & Politics

Lecture 14 — July 10, 2015
Apportionment methods

• Recall that an **apportionment method** is a function which takes as input the values \( h, n, p_1, p_2, \ldots, p_n \), where \( h \) and \( n \) are positive integers, \( p_k \)'s are positive numbers, and whose output is a sequence of nonnegative integers \( a_1, a_2, \ldots, a_n \) such that \( h = a_1 + a_2 + \ldots + a_n \).

• The interpretation is that given \( h \) objects, and \( n \) states with populations \( p_1, p_2, \ldots, p_n \), the method distributes \( a_k \) objects to the \( k^{\text{th}} \) state.

• In the case of US Congressional apportionment, \( h = 435 \) (number of seats in the House), \( n = 50 \) (number of states), and we may order the states alphabetically so that \( p_k \) is the population of the \( k^{\text{th}} \) state in alphabetical order.
Criteria and paradoxes

- An **Alabama paradox** is when a state loses a seat when \( h \) is increased and all other parameters (number of states, populations) are fixed. An apportionment method that avoids this is called **house monotone**.

- A **population paradox** is when one state gains (or remains the same) in population while another loses (or remains the same), yet it is the first state that loses a seat, while the other gains a seat. Methods that avoid this are said to be **population monotone**.

- A population paradox which results from the addition of a new state is a **new states**, or **Oklahoma, paradox**.

- We have seen that Hamilton’s method is susceptible to each of these paradoxes, and is thus neither house nor population monotone.
Quotas and divisors

• Given $h$, $n$, $p_1$, $p_2$, ..., $p_n$:

• The quantity $q_k = h(p_k/p) = p_k/(p/h)$ is the standard quota for the $k^{th}$ state.

• The lower quota is the result of rounding down $q_k$.

• The upper quota is the result of rounding up $q_k$.

• The quantity $s = p/h$ is the standard divisor. This is an ideal amount of the population entitle to each object.
Jefferson’s method

The method below, proposed by Thomas Jefferson (founding father, 3rd President) in response to Hamilton’s method, was used for apportionment in the early decades of the US.

Jefferson’s method is as follows:

1. Choose a number $d$, called the modified divisor, which represents a desired approximate size for congressional districts.
2. Compute the modified quotas $p_k/d$ for each state, and round these numbers down to obtain $a_k$.
3. If $a_1 + a_2 + \ldots + a_n = h$, then we have the apportionment.
4. Otherwise, change $d$ and try again.
Jefferson’s method (cont’d)

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- Compute the modified quotas $p_k/d$ for each state, and round these numbers down to obtain $a_k$.
- If $a_1 + a_2 + \ldots + a_n = h$, then we have the apportionment.
- Otherwise, change $d$ and try again.

In practice it is not difficult to determine an appropriate value for $d$:

- If a chosen $d$ is too large, then the sum $a_1 + a_2 + \ldots + a_n$ will be smaller than $h$.
- If a chosen $d$ is too small, then the sum $a_1 + a_2 + \ldots + a_n$ will be larger than $h$.
- For example, setting $d = s$, the standard divisor, will make $a_k$ equal to the lower quota, so $a_1 + a_2 + \ldots + a_n$ will be smaller than $h$, meaning that this divisor is too large.
- We hone in on a range of values for $d$ that will make $a_1 + a_2 + \ldots + a_n = h$. 
Jefferson’s method (cont’d)

• For example: Suppose that \( n = 3, h = 10, p_1 = 1,500,000, p_2 = 3,200,000 \) and \( p_3 = 5,300,000 \). Compute the Jefferson apportionment.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( p_k )</th>
<th>standard quota</th>
<th>Hamiton apportionment</th>
<th>Jefferson apportionment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,500,000</td>
<td>1.5</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3,200,000</td>
<td>3.2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>5,300,000</td>
<td>5.3</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>
Jefferson’s method (cont’d)

- There may be many different modified divisors $d$ which yield a Jefferson apportionment. However, these apportionments will always be the same!

**Proposition:** For $h$, $n$, $p_1$, $p_2$, ..., $p_n$ given, if $d$ and $d'$ are two different divisors yielding Jefferson apportionments $a_1$, $a_2$, ..., $a_n$ and $b_1$, $b_2$, ..., $b_n$ respectively, then $a_k = b_k$ for all $k$.

- So, as soon as we find a divisor $d$ for which $a_1 + a_2 + ... + a_n = h$ we have found *the* (one and only) Jefferson apportionment.

- R&U discuss a more systematic way to determine a good value for $d$ in Section 8.2. We will skip this.
Jefferson’s method (cont’d)

- Consider an apportionment with $n = 4$, $h = 10$, $p_1 = 1,500,000$, $p_2 = 1,400,000$, $p_3 = 1,300,000$ and $p_4 = 5,800,000$. The standard divisor is $s = 1,000,000$. Use a modified divisor of $d = 800,000$:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$p_k$</th>
<th>standard quota</th>
<th>lower quota</th>
<th>upper quota</th>
<th>Hamilton apportionment</th>
<th>modified quota</th>
<th>Jefferson apportionment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,500,000</td>
<td>1.5</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1.88</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1,400,000</td>
<td>1.4</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1.75</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1,300,000</td>
<td>1.3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1.62</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>5,800,000</td>
<td>5.8</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7.25</td>
<td>7</td>
</tr>
</tbody>
</table>

- The Jefferson apportionment to state 4 is greater than its upper quota!
Jefferson’s method (cont’d)

• Recall that an apportionment method satisfies the **quota rule** if it always apportions to each state either its upper or lower quota.

• The previous example shows that Jefferson’s method fails the quota rule.

• One can also have a **lower quota rule**, no state is assigned less than its lower quota, and an **upper quota rule**, no state is assigned more than its upper quota.

*Proposition: Jefferson’s method satisfies the lower quota rule, but not the upper quota rule.*
Jefferson’s method (cont’d)

• However, Jefferson’s method avoids the paradoxes we’ve seen with Hamilton’s method.

**Proposition:** Jefferson’s method is House monotone.

**Proposition:** Jefferson’s method is population monotone.

• We add that, generally speaking, Jefferson’s method tends to favor large states.
Other divisor methods

• Jefferson’s method is an example of a divisor method, that is, a method in which state populations are divided by a modified divisor to obtain quotas, which are then rounded (in some way), to obtain a possible apportionment.

• **Adams’ method** (named for John Quincy Adams, 6th President) is the divisor method as follows:
  • Choose a modified divisor \( d \).
  • Compute the modified quotas \( \frac{p_k}{d} \) for each state, and round these numbers up to obtain \( a_k \).
  • If \( a_1 + a_2 + \ldots + a_n = h \), then we have the desired apportionment.
  • Otherwise, change \( d \) and try again.

• Note that Adam’s method always apports at least 1 seat to each state, unlike Jefferson’s method. (We require \( h \geq n \) for Adams’.)

• Adams’ method tends to favor smaller states.
Other divisor methods (cont’d)

• **Webster’s method** (named for Daniel Webster, a senator) is the divisor method as follows:
  • Choose a modified divisor $d$.
  • Compute the modified quotas $p_k/d$ for each state, and round these numbers (in the usual way) to obtain $a_k$.
  • If $a_1 + a_2 + \ldots + a_n = h$, then we have the desired apportionment.
  • Otherwise, change $d$ and try again.

• Webster’s method was used in the late 19th and early 20th century for apportionment (often when it agreed with Hamilton’s method)
Other divisor methods (cont’d)

• For example: Suppose that $n = 3$, $h = 10$, $p_1 = 1,500,000$, $p_2 = 3,200,000$ and $p_3 = 5,300,000$. Compute the Adams and Webster apportionments.

<table>
<thead>
<tr>
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<th>$p_k$</th>
<th>standard quota</th>
<th>Jefferson apportionment</th>
<th>Adams apportionment</th>
<th>Webster apportionment</th>
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<tr>
<td>1</td>
<td>1,500,000</td>
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<td>3</td>
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</tr>
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<td>5</td>
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Geometric rounding

• We mention one last divisor method, as it is the method which has been used for congressional apportionment (and is mandated by law) since 1941.

• **Geometric rounding** is the rounding method which, given a number $x$ between integers $n$ and $n+1$, rounds $x$ to $n$ if $x < \sqrt{n(n+1)}$, and otherwise rounds $x$ to $n+1$.

• For example: If $x$ is between 1 and 2, but less than $\sqrt{2}$, then it is rounded down in this method. If $x$ is between 2 and 3, but less than $\sqrt{6}$, then it is rounded down in this method, and so on.
Hill’s method

- **Hill’s method** (or the Huntington—Hill method, named for mathematician Edward Huntington and statistician Joseph Hill) is the divisor method as follows:
  - Choose a modified divisor $d$.
  - Compute the modified quotas $p_k/d$ for each state, and round these numbers geometrically to obtain $a_k$.
  - If $a_1 + a_2 + \ldots + a_n = h$, then we have the desired apportionment.
  - Otherwise, change $d$ and try again.

- Hill’s method is often argued for on the basis that it is the *unique* apportionment method which guarantees that no additional transfer of a seat from one state to another will reduce the ratio between degrees of representation (persons per representative) in any two states.

- However, different arguments can also be given for Webster’s method.
Divisor methods

• One can show that all of these divisor methods avoid the paradoxes from last class. That is,

*Proposition:* Divisor method are house and population monotone.

• But, they may violate the quota rules. In fact:
  • Jefferson satisfies lower quota, violates upper quota.
  • Adams satisfies upper quota, violates lower quota.
  • Hill and Webster violate both quota rules.
A perfect method?

• We have seen that Hamilton’s method satisfies the quota rule, but violates population (and house) monotonicity.

• Meanwhile, divisor methods satisfy population (and house) monotonicity, but may violate the quota rules.

• Can we find a reasonable method which satisfies all of these properties?

• “Reasonable” here means neutral, that is, if state $i$ and state $j$ exchange populations (and everything else remains the same), then state $i$ and state $j$ must exchange apportioned amounts.
Balinski—Young Theorem

• Much like in the case of voting methods, our hopes are dashed by a result of the mathematicians Michel Balinski and Peyton Young:

**Theorem (Balinski—Young, 1982):** It is impossible for a neutral apportionment method to satisfy both the quota rule and population monotonicity.

• The proof, which is not too difficult, is on p. 179 of R&U.
• Recommended reading: Sections 8.1, 8.3-8.5 in R&U

• Optional reading: Anything else in Part II of R&U, particularly the proof of the Balinski—Young theorem in Section 9.6, and Ch. 12 on the History of Apportionment in the US.

• Problem set 5 has been posted on the course website, and is due on Tuesday, July 14, in class.