Turbulence and non-classifiability of projections in the Calkin algebra

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Abstract

For a Hilbert space H, the (orthogonal) projections on H are those bounded operators given by orthogonal projection onto a closed subspace of H. Two such projections are equal modulo compact if they differ by a compact operator, that is, they have the same image in the Calkin algebra, the quotient of the bounded operators by the compact operators. Using Hjorth's theory of turbulence, we show that this equivalence relation is not classifiable by countable structures, and thus there are no (countable) algebraic complete invariants for the projections in the Calkin algebra. We also analyze the complexity of equivalence modulo finite rank operators, and show that this does not admit complete invariants given by the orbits of any Polish group action. These notes were prepared for a talk given in the Logic Seminar at Cornell University, and based on the material in [1].

1 Operators and classification

Throughout, we fix an infinite dimensional separable (complex) Hilbert space H (e.g., $H = \ell^2$), with inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{B}(H)$ denote the set of all bounded linear operators on H, with the operator norm

$$\|T\| = \inf\{M > 0 : \|Tx\| \leq M\|x\| \text{ for all } x \in H\}.$$

$\mathcal{B}(H)$ is a C*-algebra under the usual operations of scalar multiplication, addition and multiplication (i.e., composition), with identity I, and adjoint operation $T \mapsto T^*$ satisfying

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

for all $x, y \in H$. Some important classes of operators in $\mathcal{B}(H)$ include:

(a) *Self-adjoint operators*: those $T \in \mathcal{B}(H)$ such that $T^* = T$;

(b) *Unitary operators*: those $U \in \mathcal{B}(H)$ such that $UU^* = U^*U = I$;

(c) *Projection operators*: those $P \in \mathcal{B}(H)$ such that $P^2 = P^* = P$, equivalently, those P given by orthogonal projection onto a closed subspace of H (namely $\mathrm{ran}(P)$). Denote the family of all projection operators by $\mathcal{P}(H)$;

(d) *Finite-rank operators*: those $T \in \mathcal{B}(H)$ such that $\mathrm{ran}(T)$ is finite dimensional. Denote the family of all finite-rank operators by $\mathcal{B}_f(H)$;

(e) *Compact operators*: those $K \in \mathcal{B}(H)$ which are (operator norm) limits of finite rank operators, equivalently, those K such that the image of the close unit ball of H under K is compact. Denote the family of all compact operators by $\mathcal{K}(H)$.

Note that $\mathcal{K}(H)$ forms a *-closed ideal in $\mathcal{B}(H)$ which is closed in the norm topology (in fact, it is the only proper such ideal), and the quotient $\mathcal{B}(H)/\mathcal{K}(H)$ is also a C*-algebra, called the *Calkin algebra*.

It is a fundamental problem in operator theory to classify families of operators in $\mathcal{B}(H)$ up to some natural notion of equivalence. The most important classical example of this is the following:

Theorem 1.1 (Weyl–von Neumann). *If S and T are self-adjoint operators in $\mathcal{B}(H)$, then S and T are* unitarily equivalent modulo compact *(i.e., there is a unitary operator U and a compact operator K such that $USU^* - K = T$) if and only if S and T have the same* essential spectrum *(i.e., the spectrum of their images in the Calkin algebra).*

We will see below that the theory of *Borel equivalence relations* provides both a general framework for discussing such a classification, and powerful tools for showing when such a classification is impossible.

2 Borel equivalence relations and turbulence

Recall that a *Polish space* is a completely metrizable separable space, e.g., $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, separable Banach spaces, etc, and countable products and closed (or G_δ) subspaces of these. A *standard Borel space* is a measurable space X with σ-algebra $\mathcal{B}$, where $\mathcal{B}$ is the collection of Borel subsets of some (unspecified) Polish topology on X; a standard Borel space is a Polish space which has forgotten its topology but remembered its Borel structure.

We say that an equivalence relation E on a Polish (or standard Borel) space X is *Borel* if the set $E = \{(x,y) \in X^2 : xEy\}$ is Borel as a subset of X^2.

If E and F are equivalence relations on Polish (or standard Borel) spaces X and Y respectively, a map $f : X \to Y$ is called a *Borel reduction* of E to F if f is a Borel function (i.e., inverse images of open sets in Y are Borel in X), and

$$xEy \iff f(x)Ff(y)$$

for all $x, y \in X$. In this case, we say that E is *Borel reducible* to F, and write $E \leq_B F$. If $E \leq_B F$ and $F \leq_B E$, then we say that E and F are *Borel bireducible*, and write $E \equiv_B F$.

Conceptually, the existence of a Borel reduction from E to F shows that F-equivalent objects can be viewed as definable *complete invariants* for E-equivalent objects, and by composing reductions, complete invariants for F yield complete invariants for E. Thus, $\leq_B$ is a measure of difficulty of complete classification.

The following are important examples of Borel equivalence relations, relevant to our results below.

Example 2.1. If X is a Polish space, we denote by $\Delta(X)$ the equality relation on X. Clearly $\Delta(X)$ is a closed, and thus Borel, subset of X^2.

Example 2.2. We identify $2 = \{0,1\}$, having the discrete topology. The Borel equivalence relation E_0 is defined on $2^{\mathbb{N}}$ by

$$(x_n)_n E_0 (y_n)_n \iff \exists m \forall n \geq m (x_n = y_n).$$

Example 2.3. The Borel equivalence relation E_1 is defined on $[0,1]^{\mathbb{N}}$ by

$$(x_n)_n E_1 (y_n)_n \iff \exists m \forall n \geq m (x_n = y_n).$$

Example 2.4. The Borel equivalence relation $E = \mathbb{R}^{\mathbb{N}}/c_0$ is defined on $\mathbb{R}^{\mathbb{N}}$ by

$$(x_n)_n E (y_n)_n \iff \lim_{n \to \infty} |x_n - y_n| = 0.$$

Its restriction $[0,1]^{\mathbb{N}}/c_0$ to $[0,1]^{\mathbb{N}}$ is defined similarly. Note that $\mathbb{R}^{\mathbb{N}}/c_0$ is exactly the orbit equivalence relation of the translation action of the subspace c_0 on $\mathbb{R}^{\mathbb{N}}$.

The simplest Borel equivalence relations, called *smooth*, are those which are Borel reducible to $\Delta(Y)$ for some Polish space Y. Since all uncountable Polish spaces are Borel isomorphic, smooth equivalence relations are exactly those which admit complete classification by real numbers.

The following important theorem, the *Glimm–Effros Dichotomy*, shows that the equivalence relation E_0 is the canonical obstruction to such classification.

Theorem 2.1 (Harrington–Kechris–Louveau)**.** *For a Borel equivalence relation E, exactly one of the following holds:*

(i) *E is smooth.*

(ii) *$E_0 \leq_B E$.*

A *Polish group* G is a topological group which has a Polish topology. Recall that an action of a Polish group G on a Polish space X is continuous if the map $G \times X \to X : (g, x) \mapsto g \cdot x$ is continuous (for such groups and spaces, this is equivalent to separate continuity). Given such an action on X, we can associate to it the orbit equivalence relation E_G given by

$$x \, E_G \, y \iff \exists g \in G (g \cdot x = y).$$

This equivalence relation is not Borel in general (it is always *analytic*, a continuous image of a Borel set), but is in many interesting cases, and for large classes of groups (e.g., countable discrete groups and locally compact groups). In particular, if X itself is a Polish group and G a Borel subgroup of X which can be given a Polish group topology with the same Borel sets (i.e., G is *Polishable*), then the translation action of G on X is continuous and E_G is Borel.

The following theorem, one half of an open conjecture, shows that E_1 is an obstruction to classification by orbits of Polish group actions.

Theorem 2.2 (Kechris–Louveau)**.** *Let G be a Polish group acting continuously on a Polish space X. Then, $E_1 \not\leq_B E_G$.*

Orbit equivalence relations of Polish group actions can be used to model the isomorphism relation on the class of countable structures of a first-order theory, e.g., groups, rings, graphs, etc. If a Borel equivalence relation E is Borel reducible to such a relation, then we say that E is *classifiable by countable structures.* For instance, using countable structures with countably many unary predicates, each one corresponding to the cut below some $q \in \mathbb{Q}$, one can show that $\Delta(\mathbb{R})$ is classifiable by countable structures.

Hjorth isolated a dynamical property of such actions, called *turbulence*, which implies that the corresponding orbit equivalence relation resists classification by countable structures.

Let G be a Polish group acting continuously on a Polish space X. For $U \subseteq X$ open, and $V \subseteq G$ a symmetric open neighborhood of the identity e_G, the (U,V)-*local graph* of the action is defined by

$$xR_{U,V}y \quad \Longleftrightarrow \quad x, y \in U \text{ and } \exists g \in V(g \cdot x = y).$$

The (U,V)-*local orbit* of a point $x \in U$, denoted by $\mathcal{O}(x,U,V)$, is the (graph theoretic) connected component of x in $R_{U,V}$.

For such an X and G, we say that the action of G is *turbulent* if

(i) every orbit is dense,

(ii) every orbit is meager,

(iii) every (U,V)-local orbit is *somewhere dense*, i.e., for every U and V as above, and every $x \in U$, $\overline{\mathcal{O}(x,U,V)}$ has nonempty interior.

The action is *generically turbulent* if it is turbulent when restricted to an invariant dense G_δ set.

Theorem 2.3 (Hjorth). *Let G be a Polish group acting continuously on a Polish space X. If the action of G is generically turbulent, then E_G is not classifiable by countable structures.*

Consequently, if E_G is as in the theorem, and E a Borel equivalence relation for which $E_G \leq_B E$, then E is not classifiable by countable structures. The following are important examples of turbulent actions:

Example 2.5. We say that a subgroup G of the additive group $\mathbb{R}^{\mathbb{N}}$ is *strongly dense* if for every finite sequence $(x_0, \ldots, x_n)$ of real numbers, there is a $y = (y_0, y_1, \ldots) \in G$ such that $y_i = x_i$ for $0 \leq i \leq n$. Note that, in particular, this implies that G is dense in the product topology on $\mathbb{R}^{\mathbb{N}}$.

If G is a proper, Polishable, and strongly dense subgroup of $\mathbb{R}^{\mathbb{N}}$, then the translation action of G on $\mathbb{R}^{\mathbb{N}}$ is turbulent. We denote the corresponding equivalence relation by $\mathbb{R}^{\mathbb{N}}/G$. Examples of such subgroups are c_0 and ℓ^p for $1 \leq p < \infty$.

Example 2.6. Let X be a separable Frechet space, i.e., a Polish locally convex topological (real or complex) vector space. If Y is a proper, Polishable, dense subspace of X, then the action of Y on X by translation is turbulent. Examples of such pairs (X,Y) include those described in the previous example, as well as $(C([0,1]), C^\infty([0,1]))$, $(L^p([0,1]), C([0,1]))$ and (c_0, ℓ^p) for $1 \leq p < \infty$.

Example 2.7. Let $\mathbb{T}$ be the unit circle $\{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\}$. As in the case of $\mathbb{R}^{\mathbb{N}}$, we say that a subgroup G of $\mathbb{T}^{\mathbb{N}}$ is *strongly dense* if for all finite sequences $(z_0, \ldots, z_n)$ of unit complex numbers, there is a $g = (g_0, g_1, \ldots) \in G$ such that $g_i = z_i$ for $0 \leq i \leq n$. Again, strong density implies density.

If G is a proper, Polishable, and strongly dense subgroup of $\mathbb{T}^{\mathbb{N}}$, then the translation action of G on $\mathbb{T}^{\mathbb{N}}$ is turbulent. The proof of this following fact can be modeled on the corresponding result for strongly dense subgroups of $\mathbb{R}^{\mathbb{N}}$. An important such subgroup is given by

$$G_0 = \operatorname{ran}(\varphi_{c_0}) = \{z \in \mathbb{T}^{\mathbb{N}} : \lim_n z_n = 1\},$$

in which case the action of G_0 by translation on $\mathbb{T}^{\mathbb{N}}$ is orbit equivalent to the action of c_0 on $\mathbb{T}^{\mathbb{N}}$ given by

$$(\alpha_n)_n \cdot (e^{i\theta_n})_n = e^{i(\theta_n + \alpha_n)},$$

for $(\alpha_n)_n$ in c_0.

Using the turbulence of the action of G_0 on $\mathbb{T}^{\mathbb{N}}$, one can establish the fact that $[0,1]^{\mathbb{N}}/c_0$ is not classifiable by countable structures. In fact,

Theorem 2.4. *If $\mathbb{T}^{\mathbb{N}}/G_0$ denotes the orbit equivalence relation of the translation action of G_0 on $\mathbb{T}^{\mathbb{N}}$, then $\mathbb{T}^{\mathbb{N}}/G_0$ is Borel bireducible with $[0,1]^{\mathbb{N}}/c_0$.*

Proof. (Sketch.) First, one shows that the inclusion of $\mathbb{T}^{\mathbb{N}}$ into $([-1,1]^2)^{\mathbb{N}}$ is a Borel reduction of equivalence modulo $\mathbb{T}^{\mathbb{N}}/G_0$ to $([-1,1]^2)^{\mathbb{N}}/c_0 \times c_0$. The latter is clearly Borel isomorphic to $([0,1]^2)^{\mathbb{N}}/c_0 \times c_0$, which is Borel reducible to $[0,1]^{\mathbb{N}}/c_0$ via the map which alternates coordinates. Lastly, $[0,1]^{\mathbb{N}}/c_0$ is Borel reducible to $\mathbb{T}^{\mathbb{N}}/G_0$ via the map $[0,1]^{\mathbb{N}} \to \mathbb{T}^{\mathbb{N}} : (\alpha_n)_n \mapsto (e^{i\pi/2(\alpha_n)})_n$. □

3 Borel structures on classes of operators

To cast the problem of classification for families of operators in this setting, we need to verify that the underlying family of operators has a meaningful Polish topology or standard Borel structure, and that the equivalence relations studied are Borel. Unfortunately, the norm topology on $\mathcal{B}(H)$ is not even separable, so we instead use the *strong operator topology*, whose basic open sets are given by

$$U = \{S \in \mathcal{B}(H) : \|(S-T)x_0\| < \epsilon \wedge \ldots \wedge \|(S-T)x_n\| < \epsilon\},$$

for $T \in \mathcal{B}(H)$, $x_0, \ldots, x_n \in H$ and $\epsilon > 0$. Denote by $\mathcal{B}(H)_{\leq 1} = \{T \in \mathcal{B}(H) : \|T\| \leq 1\}$. The following facts are well-known, and can be easily shown:

Proposition 3.1. (a) *$\mathcal{B}(H)_{\leq 1}$ is Polish in the strong operator topology.*

(b) *$\mathcal{B}(H)$ is a standard Borel space, with Borel structure inherited from the strong operator topology (it is not a Polish space).*

(c) *The collections of self-adjoint, unitary, and projection operators in $\mathcal{B}(H)_{\leq 1}$ are Polish in the strong operator topology.*

(d) *$\mathcal{B}_f(H)$ is a Borel subset of $\mathcal{B}(H)$ in the strong operator topology.*

(e) *$\mathcal{K}(H)$ is Polish in the norm topology, and a Borel subset of $\mathcal{B}(H)$ in the strong operator topology.*

One can show that unitary equivalence modulo compact is a Borel equivalence relation on the standard Borel space of all self-adjoint operators, and it has been shown recently that the Weyl–von Neumann theorem actually expresses the smoothness of this equivalence relation.

Theorem 3.1 (Ando–Matsuzawa)**.** *The map $T \mapsto \sigma_{\mathrm{ess}}(T)$ is a Borel function from the space of bounded self-adjoint operators to the Effros Borel space of closed subsets of $\mathbb{R}$. In particular, unitary equivalence modulo compact of bounded self-adjoint operators is smooth.*

The equivalence relations we will focus on are as follows:

$$\begin{aligned} T \equiv_{\mathrm{ess}} S \quad &\iff \quad T - S \in \mathcal{K}(H)\,(\textit{modulo compact} \text{ or } \textit{essential equivalence}) \\ T \equiv_f S \quad &\iff \quad T - S \in \mathcal{B}_f(H) \quad (\textit{modulo finite rank}) \end{aligned}$$

We are particularly interested in the restrictions of these equivalence relations to the Polish space of projections $\mathcal{P}(H)$, though first we give an application of the theory of turbulence to the restriction of $\equiv_{\mathrm{ess}}$ to $\mathcal{B}(H)_{\leq 1}$. Fix an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ for H, and consider the map $[0,1]^{\mathbb{N}} \to \mathcal{B}(H)_{\leq 1} : \alpha \mapsto T_\alpha$ given by the formula

$$T_\alpha x = \sum_{n=0}^{\infty} \alpha_n \langle x, e_n \rangle e_n,$$

for $\alpha = (\alpha_n)_n \in [0,1]^{\mathbb{N}}$ and $x \in H$. It is clear that $\|T_\alpha\| \leq 1$ for all $\alpha \in [0,1]^{\mathbb{N}}$.

Lemma 3.1. *The map $[0,1]^{\mathbb{N}} \to \mathcal{B}(H)_{\leq 1} : \alpha \mapsto T_\alpha$ is continuous with respect to the strong operator topology, and in particular, is Borel.*

Proof. It is clear that map is injective. Fix $\alpha \in [0,1]^{\mathbb{N}}$ and let $U = \{T \in \mathcal{B}(H)_{\leq 1} : \|(T - T_\alpha)v\| < \epsilon\}$ be a subbasic open neighborhood of T_α, where $v = \sum_{n=0}^{\infty} a_n e_n \in H$ and $\epsilon > 0$. Pick m such that $\sum_{n=m+1}^{\infty} |a_n|^2 < \epsilon^2/2$, and let

$$V = \left\{ \beta \in [0,1]^{\mathbb{N}} : \sum_{n=0}^{m} |\beta_n - \alpha_n|^2 |a_n|^2 < \epsilon^2/2 \right\}.$$

It is clear that V is an open neighborhood of α in $[0,1]^{\mathbb{N}}$. If $\beta \in V$, then

$$\|(T_\beta - T_\alpha)v\|^2 = \sum_{n=0}^{\infty} |\beta_n - \alpha_n|^2 |a_n|^2 = \sum_{n=0}^{m} |\beta_n - \alpha_n|^2 |a_n|^2 + \sum_{n=m+1}^{\infty} |\beta_n - \alpha_n|^2 |a_n|^2 < \epsilon^2,$$

showing that V is contained in the preimage of U. Thus, the map is continuous. $\square$

Proposition 3.2. $[0,1]^{\mathbb{N}}/c_0 \leq_B \equiv_{\mathrm{ess}}$ *restricted to $\mathcal{B}(H)_{\leq 1}$.*

Proof. We use the map $[0,1]^{\mathbb{N}} \to \mathcal{B}(H)_{\leq 1} : \alpha \mapsto T_\alpha$ defined above. Suppose that $\alpha, \beta \in [0,1]^{\mathbb{N}}$. For $x \in H$, we have that

$$(T_\alpha - T_\beta)v = \sum_{n=0}^{\infty} (\alpha_n - \beta_n) \langle x, e_n \rangle e_n.$$

By a well-known characterization of compactness for operators which are diagonal with respect to a given orthonormal basis, $T_\alpha - T_\beta$ is compact if and only if $\alpha - \beta \in c_0$. $\square$

Corollary 3.1. $\equiv_{\mathrm{ess}}$ *(on $\mathcal{B}(H)$ or restricted to $\mathcal{B}(H)_{\leq 1}$) is not classifiable by countable structures.*

A more sophisticated example of such a result is the following, which uses the turbulence of the action of ℓ^2 on $\mathbb{R}^{\mathbb{N}}$ and a classical result of Kakutani.

Theorem 3.2 (Kechris–Sofronidis)**.** *Unitary equivalence of self-adjoint (or unitary) operators is not classifiable by countable structures.*

In fact, Kechris–Sofronidis also show that the action of the unitary group by conjugation on the self-adjoint operators, or on itself, is generically turbulent.

4 Projections modulo compact

Throughout, we consider $\mathcal{P}(H)$ as a Polish space in the strong operator topology. There are several natural notions of equivalence on $\mathcal{P}(H)$, particularly in light of its identification with the lattice of closed subspaces of H, but we will focus on the restrictions of $\equiv_{\text{ess}}$ and $\equiv_f$. We note that, although a projection is compact if and only if it is finite rank, this is not true of the *difference* of two projections. In particular, $\equiv_{\text{ess}}$ does not coincide with $\equiv_f$ on $\mathcal{P}(H)$.

The image of a projection under the quotient mapping onto the Calking algebra $\mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$ remains a projection (i.e., a self-adjoint idempotent), and two projections have the same image under this map if and only if they are $\equiv_{\text{ess}}$-equivalent. Moreover, one can show using spectral theory that *every* projection in $\mathcal{B}(H)/\mathcal{K}(H)$ is the image of a projection in $\mathcal{B}(H)$, and thus, the quotient of $\mathcal{P}(H)$ by $\equiv_{\text{ess}}$ is exactly the set of projections in the Calkin algebra. For this reason, the results which follow can be described as (non-)classification results for projections in Calkin algebra.

Fix an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ for H. For each $x \subseteq \mathbb{N}$, let P_x be the projection onto the subspace $\overline{\text{span}}\{e_n : n \in x\}$. Observe that, for $v \in H$,

$$P_x v = \sum_{n=0}^{\infty} x_n \langle v, e_n \rangle e_n = \sum_{n \in x} \langle v, e_n \rangle e_n,$$

where x is identified with its characteristic sequence $(x_n)_n$, with $x_n = 1$ if $n \in x$ and $x_n = 0$ otherwise. The map $2^{\mathbb{N}} \to \mathcal{P}(H) : x \mapsto P_x$ is called the *diagonal embedding* with respect to the orthonormal basis $\{e_n : n \in \mathbb{N}\}$.

Proposition 4.1. *The diagonal embedding $2^{\mathbb{N}} \to \mathcal{P}(H) : x \mapsto P_x$ is a continuous injection.*

Proof. It is clear that the map $x \mapsto P_x$ is an injection. Let $x \in 2^{\mathbb{N}}$, and let $U = \{P \in \mathcal{P}(H) : \|(P - P_x)v\| < \epsilon\}$ be a subbasic open neighborhood of P_x in $\mathcal{P}(H)$, where $\epsilon > 0$ and $v = \sum_{n=0}^{\infty} a_n e_n \in H$. Let $m \in \mathbb{N}$ be such that $\sum_{n=m}^{\infty} |a_n|^2 < \epsilon^2$. Consider the open neighborhood of x in $2^{\mathbb{N}}$ given by

$$V = \{y \in 2^{\mathbb{N}} : \forall n < m (y_n = x_n)\}.$$

Given $y \in V$, we have

$$\begin{aligned} \|(P_y - P_x)v\|^2 &= \left\| (P_y - P_x) \left(\sum_{n=m}^{\infty} a_n e_n \right) \right\|^2 \\ &\leq \left\| \sum_{n=m}^{\infty} a_n e_n \right\|^2 = \sum_{n=m}^{\infty} |a_n|^2 < \epsilon^2, \end{aligned}$$

and so $P_y \in V$. Thus, the map $x \mapsto P_x$ is continuous. □

Proposition 4.2. *$E_0 \leq_B \equiv_{\text{ess}}$ restricted to $\mathcal{P}(H)$.*

Proof. Fix an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ and let $2^{\mathbb{N}} \to \mathcal{P}(H) : x \mapsto P_x$ be the diagonal embedding. We have already seen that this is a continuous injection. For $x, y \in 2^{\mathbb{N}}$, observe that

$$(P_x - P_y)v = \sum_{n=0}^{\infty} (x_n - y_n) \langle v, e_n \rangle e_n$$

for all $v \in H$. As above, this diagonal operator is compact if and only if $x_n - y_n \to 0$, but since $x_n - y_n \in \{-1, 0, 1\}$ for all n, this occurs if and only if $x_n = y_n$ for all but finitely many n. Thus xE_0y if and only if $P_x \equiv_{\text{ess}} P_y$, showing that the map is a reduction. □

Consequently, there can be no hope for a smooth classification, a la the Weyl–von Neumann theorem, result for projections modulo compact. However, the previous proof also shows that $\equiv_{\text{ess}}$ when restricted to projections diagonal with respect to the same orthonormal basis is Borel isomorphic to E_0, and thus such a "diagonal" reduction can never show that $\equiv_{\text{ess}}$ is not classifiable by countable structures. We need a noncommutative "twist".

We define a map $[0,1]^{\mathbb{N}} \to \mathcal{P}(H) : \alpha \mapsto P_\alpha$ as follows: Fix an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ for H. For each $\alpha = (\alpha_n)_n \in [0,1]^{\mathbb{N}}$, let P_α be the projection onto the subspace $\overline{\text{span}}\{e_{2n} + \alpha_n e_{2n+1} : n \in \mathbb{N}\}$. This is the first map into the space of operators that we have considered whose range is not simultaneously diagonalizable with respect to a fixed basis, nor is the range commutative. Observe that $\text{ran}(P_\alpha)$ has an orthonormal basis given by

$$\left\{ \frac{1}{\sqrt{1+\alpha_n^2}}(e_{2n} + \alpha_n e_{2n+1}) : n \in \mathbb{N} \right\},$$

and thus we can write, for $v = \sum_{n=0}^{\infty} a_n e_n \in H$,

$$\begin{aligned}
P_\alpha v &= \sum_{n=0}^{\infty} \left\langle \sum_{k=0}^{\infty} a_k e_k, \frac{1}{\sqrt{1+\alpha_n^2}}(e_{2n} + \alpha_n e_{2n+1}) \right\rangle \frac{1}{\sqrt{1+\alpha_n^2}}(e_{2n} + \alpha_n e_{2n+1}) \\
&= \sum_{n=0}^{\infty} \frac{1}{1+\alpha_n^2} \langle a_{2n} e_{2n} + a_{2n+1} e_{2n+1}, e_{2n} + \alpha_n e_{2n+1} \rangle (e_{2n} + \alpha_n e_{2n+1}) \\
&= \sum_{n=0}^{\infty} \frac{a_{2n} + a_{2n+1}\alpha_n}{1+\alpha_n^2}(e_{2n} + \alpha_n e_{2n+1}).
\end{aligned}$$

Since we must consider the operator $P_\alpha - P_\beta$ several times in the proofs that follow, it will be useful to put it into a canonical form. Let $\alpha, \beta \in [0,1]^{\mathbb{N}}$ and $v \in H$ be as above, then

$$\begin{aligned}
(P_\alpha - P_\beta)v &= \sum_{n=0}^{\infty} \frac{a_{2n} + a_{2n+1}\alpha_n}{1+\alpha_n^2}(e_{2n} + \alpha_n e_{2n+1}) \\
&\quad - \sum_{n=0}^{\infty} \frac{a_{2n} + a_{2n+1}\beta_n}{1+\beta_n^2}(e_{2n} + \beta_n e_{2n+1}) \\
&= \sum_{n=0}^{\infty} \left[\frac{a_{2n} + a_{2n+1}\alpha_n}{1+\alpha_n^2} - \frac{a_{2n} + a_{2n+1}\beta_n}{1+\beta_n^2} \right] e_{2n} \\
&\quad + \sum_{n=0}^{\infty} \left[\frac{a_{2n}\alpha_n + a_{2n+1}\alpha_n^2}{1+\alpha_n^2} - \frac{a_{2n}\beta_n + a_{2n+1}\beta_n^2}{1+\beta_n^2} \right] e_{2n+1}.
\end{aligned}$$

Denote by T_0, T_1, T_2 and T_3 the operators

$$T_0 v = \sum_{n=0}^{\infty} \left[\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2} \right] a_{2n} e_{2n},$$
$$T_1 v = \sum_{n=0}^{\infty} \left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2} \right] a_{2n+1} e_{2n+1},$$

$$T_2 v = \sum_{n=0}^{\infty} \left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2} \right] a_{2n} e_{2n},$$
$$T_3 v = \sum_{n=0}^{\infty} \left[\frac{\alpha_n^2}{1+\alpha_n^2} - \frac{\beta_n^2}{1+\beta_n^2} \right] a_{2n+1} e_{2n+1},$$

and by S_0 and S_1 the operators

$$S_0 v = \sum_{n=0}^{\infty} a_{2n+1} e_{2n},$$
$$S_1 v = \sum_{n=0}^{\infty} a_{2n} e_{2n+1},$$

Each of the operators T_0, T_1, T_2, T_3, S_0 and S_1 is clearly bounded, with $\|S_0\| \leq 1$ and $\|S_1\| \leq 1$. By collecting terms, one can show that

$$P_\alpha - P_\beta = T_0 + S_0 T_1 + S_1 T_2 + T_3.$$

Lemma 4.1. *The map $[0,1]^{\mathbb{N}} \to \mathcal{P}(H) : \alpha \mapsto P_\alpha$ is a continuous injection, and in particular, is Borel.*

Theorem 4.1. $[0,1]^{\mathbb{N}}/c_0 \sqsubseteq_B \equiv_{\text{ess}}$ *restricted to $\mathcal{P}(H)$.*

Proof. It remains to show that this map is a reduction of $[0,1]^{\mathbb{N}}/c_0$ to $\equiv_{\text{ess}}$. Suppose that $\alpha, \beta \in [0,1]^{\mathbb{N}}$, and $\alpha - \beta \in c_0$. For $v = \sum_{n=0}^{\infty} a_n e_n$, we use the inequalities

$$\left| \frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2} \right| = \left| \frac{\beta_n^2 - \alpha_n^2}{(1+\alpha_n^2)(1+\beta_n^2)} \right| \leq |\beta_n - \alpha_n||\beta_n + \alpha_n|$$
$$\left| \frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2} \right| = \left| \frac{\alpha_n + \alpha_n\beta_n^2 - \beta_n - \alpha_n^2\beta_n}{(1+\alpha_n^2)(1+\beta_n^2)} \right|$$
$$\leq |\alpha_n - \beta_n| + |\alpha_n||\beta_n - \alpha_n||\beta_n|,$$
$$\left| \frac{\alpha_n^2}{1+\alpha_n^2} - \frac{\beta_n^2}{1+\beta_n^2} \right| = \left| \frac{\alpha_n^2 - \beta_n^2}{(1+\alpha_n^2)(1+\beta_n^2)} \right| \leq |\beta_n - \alpha_n||\beta_n + \alpha_n|,$$

and the aforementioned characterization of diagonal compact operators, to see that T_0, T_1, T_2 and T_3 are compact. Since the compact operators form an ideal, $S_0 T_1$ and $S_1 T_2$ are also compact, and thus so is $P_\alpha - P_\beta$.

Conversely, take $\alpha, \beta \in [0,1]^{\mathbb{N}}$ and suppose that $P_\alpha - P_\beta$ is compact. We will use that if an operator is compact, then it is *weak–norm continuous* on the closed unit ball of H. Since the sequence e_m converges *weakly* to 0 as $m \to \infty$, i.e., for each $y \in H$, $\langle e_m, y\rangle \to 0$ as $m \to \infty$, it follows that $(P_\alpha - P_\beta)e_{2m}$ and $(P_\alpha - P_\beta)e_{2m+1}$ converge *in norm* to 0. Observe that

$$(P_\alpha - P_\beta)e_{2m} = \left[\frac{1}{1+\alpha_m^2} - \frac{1}{1+\beta_m^2}\right] e_{2m} + \left[\frac{\alpha_m}{1+\alpha_m^2} - \frac{\beta_m}{1+\beta_m^2}\right] e_{2m+1},$$
$$(P_\alpha - P_\beta)e_{2m+1} = \left[\frac{\alpha_m}{1+\alpha_m^2} - \frac{\beta_m}{1+\beta_m^2}\right] e_{2m} + \left[\frac{\alpha_m^2}{1+\alpha_m^2} - \frac{\beta_m^2}{1+\beta_m^2}\right] e_{2m+1}.$$

Thus,

$$\|(P_\alpha - P_\beta)e_{2m}\|^2 = \left|\frac{1}{1+\alpha_m^2} - \frac{1}{1+\beta_m^2}\right|^2 + \left|\frac{\alpha_m}{1+\alpha_m^2} - \frac{\beta_m}{1+\beta_m^2}\right|^2,$$
$$\|(P_\alpha - P_\beta)e_{2m+1}\|^2 = \left|\frac{\alpha_m}{1+\alpha_m^2} - \frac{\beta_m}{1+\beta_m^2}\right|^2 + \left|\frac{\alpha_m^2}{1+\alpha_m^2} - \frac{\beta_m^2}{1+\beta_m^2}\right|^2$$

and both converge to 0 as $m \to \infty$. We also have the inequalities

$$\left|\frac{1}{1+\alpha_m^2} - \frac{1}{1+\beta_m^2}\right| = \left|\frac{\beta_m^2 - \alpha_m^2}{(1+\alpha_m^2)(1+\beta_m^2)}\right| \geq \frac{1}{4}|\alpha_m - \beta_m||\alpha_m + \beta_m|,$$
$$\left|\frac{\alpha_m}{1+\alpha_m^2} - \frac{\beta_m}{1+\beta_m^2}\right| = \left|\frac{\alpha_m + \alpha_m\beta_m^2 - \beta_m - \alpha_m^2\beta_m}{(1+\alpha_m^2)(1+\beta_m^2)}\right|$$
$$\geq \frac{1}{4}|\alpha_m - \beta_m||1 - \alpha_m\beta_m|,$$

and so the quantities on the right hand side must also converge to 0. For any m, since $\alpha_m, \beta_m \in [0,1]$, we have $\alpha_m + \beta_m \geq \sqrt{2\alpha_m\beta_m} \geq \alpha_m\beta_m$ and so

$$|\alpha_m + \beta_m| + |1 - \alpha_m\beta_m| = \alpha_m + \beta_m + 1 - \alpha_m\beta_m \geq 1.$$

Thus,

$$|\alpha_m - \beta_m||\alpha_m + \beta_m| + |\alpha_m - \beta_m||1 - \alpha_m\beta_m| \geq |\alpha_m - \beta_m|,$$

showing that $\alpha_m - \beta_m$ converges to 0, i.e., $\alpha - \beta \in c_0$, as claimed. □

Corollary 4.1. $\equiv_{\mathrm{ess}}$ *restricted to* $\mathcal{P}(H)$ *is not classifiable by countable structures.*

5 Projections modulo finite rank

There are two natural ways to define equivalence modulo "finite rank" or "finite dimensions" on $\mathcal{P}(H)$. The first is to simply restrict the equivalence relation $\equiv_f$, induced by the finite-rank operators $\mathcal{B}_f(H)$, to $\mathcal{P}(H)$. The second is to say that $P \equiv_{fd} Q$ if $\mathrm{ran}(P)$ is contained in a finite-dimensional extension of $\mathrm{ran}(Q)$, and vice-versa. That is, $P \equiv_{fd} Q$ if and only if there exists vectors $v_0, \ldots, v_n, u_0, \ldots, u_m \in H$ such that $\mathrm{ran}(P) \subseteq \overline{\mathrm{span}}(\mathrm{ran}(Q) \cup \{v_0, \ldots, v_n\})$ and $\mathrm{ran}(Q) \subseteq \overline{\mathrm{span}}(\mathrm{ran}(P) \cup \{u_0, \ldots, u_m\})$. In fact, these notions coincide.

Lemma 5.1. *Let* $P, Q \in \mathcal{P}(H)$*. The following are equivalent:*

(i) $P \equiv_{fd} Q$.

(ii) *There exists mutually orthonormal vectors* $w_0, \ldots, w_k \in \mathrm{ran}(P)^\perp$ *and* $y_0, \ldots, y_l \in \mathrm{ran}(Q)^\perp$ *such that*

$$\overline{\mathrm{span}}(\mathrm{ran}(P) \cup \{w_0, \ldots, w_k\}) = \overline{\mathrm{span}}(\mathrm{ran}(Q) \cup \{y_0, \ldots, y_l\}).$$

(iii) *There exists finite rank projections R and R' with $RP = 0$, $R'Q = 0$ and $P + R = Q + R'$.*

(iv) $P \equiv_f Q$.

In light of the previous proposition, we will use $\equiv_f$ for this relation. As in the case of $\equiv_{\text{ess}}$, the diagonal embedding witnesses the non-smoothness of $\equiv_f$ on $\mathcal{P}(H)$.

Proposition 5.1. *$E_0 \leq_B \equiv_f$ restricted to $\mathcal{P}(H)$.*

Proof. This proof is exactly as in the case for $\equiv_{\text{ess}}$. □

We will use the same map $[0,1]^{\mathbb{N}} \to \mathcal{P}(H) : \alpha \mapsto P_\alpha$ as in the proof of $[0,1]^{\mathbb{N}}/c_0 \leq_B; \equiv_{\text{ess}}$ on $\mathcal{P}(H)$, and show that it is also a reduction of E_1 to $\equiv_f$.

Theorem 5.1. *$E_1 \sqsubseteq_B \equiv_f$ restricted to $\mathcal{P}(H)$.*

Proof. As above, for $\alpha, \beta \in [0,1]^{\mathbb{N}}$ and $v = \sum_{n=0}^{\infty} a_n e_n \in H$,

$$P_\alpha - P_\beta = T_0 + S_0 T_1 + S_1 T_2 + T_3.$$

Clearly, if $\alpha E_1 \beta$, then all but finitely many of the coefficients (which are independent of v) $\left[\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2}\right]$, $\left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2}\right]$ and $\left[\frac{\alpha_n^2}{1+\alpha_n^2} - \frac{\beta_n^2}{1+\beta_n^2}\right]$ will be 0, showing that $P_\alpha - P_\beta$ has finite rank.

Conversely, suppose that $P_\alpha - P_\beta$ has finite rank. It follows that the operator $T = T_0 + S_0 T_1$, given by

$$Tv = \sum_{n=0}^{\infty} \left[\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2}\right] a_{2n} e_{2n} + \sum_{n=0}^{\infty} \left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2}\right] a_{2n+1} e_{2n}$$

for $v = \sum_{n=0}^{\infty} a_n e_n$, is of finite rank. Using vectors of the form $\sum_{n=0}^{\infty} a_{2n} e_{2n}$ and $\sum_{n=0}^{\infty} a_{2n+1} e_{2n+1}$ it is easy to see that in order for T to be finite rank, all but finitely many of the terms $\left[\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2}\right]$, and $\left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2}\right]$ are 0. Since $\alpha_n \geq 0$ and $\beta_n \geq 0$, $\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2} = 0$ if and only if $\alpha_n = \beta_n$. Thus, $\alpha E_1 \beta$, showing that the map is a reduction. □

Corollary 5.1. *$\equiv_f$ restricted to $\mathcal{P}(H)$ is not Borel reducible to the orbit equivalence relation of any Polish group action.*

References

[1] I. B. Smythe. Borel equivalence relations in the space of bounded operators. *arXiv:1407.5325 [math.LO]*, 2014.