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MATH 2940 (Dis. 204, 208, 216) – Quiz #1

Answer the following questions in the space provided. (There is a question on the back!)
Justify all answers.

Problem 1. For which values of h is the following linear system consistent?

$$-4x_1 + 12x_2 = h$$
$$2x_1 - 6x_2 = -3$$

Solution. Consider the corresponding augmented matrix to this linear system:

$$\left(\begin{array}{cc|c} -4 & 12 & h \\ 2 & -6 & -3 \end{array}\right) \xrightarrow{R_2+\frac{1}{2}R_1} \left(\begin{array}{cc|c} -4 & 12 & h \\ 0 & 0 & -3+\frac{1}{2}h \end{array}\right)$$

Thus, the system above is consistent if and only if $h = 6$. □

Problem 2.
Can $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$ be expressed as a linear combination of $\mathbf{a}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{a}_3 = \begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}$?

Solution. The problem is asking if we can find real numbers (weights) x_1, x_2, x_3 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

This is equivalent to determining whether the linear system with augmented matrix given below is consistent.

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right) \xrightarrow{R_2+2R_1} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array}\right) \xrightarrow{R_3-2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since the echelon form obtained on the right does not contain a row of the form

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & b \end{array}\right), \quad \text{where } b \neq 0,$$

the system is consistent. Thus, $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$. (In fact, since this system has infinitely many solutions, there are infinitely many ways that $\mathbf{b}$ can be expressed as such a linear combination, though this not relevant here.) □

Problem 3. True or False: If A is an $m \times n$ matrix, does the matrix equation

$$A\mathbf{x} = \mathbf{0}, \quad \text{where} \quad \mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \text{ in } \mathbb{R}^n,$$

always have a solution $\mathbf{x}$? Justify your answer.

Solution. [Minor typo: it should say that $\mathbf{0}$ is in $\mathbb{R}^m$. Also, apologies for not technically asking a "true or false" question...]

True: $\mathbf{x} = \mathbf{0}$ in $\mathbb{R}^n$, i.e., an n-tuple of 0s, is always a solution (called the *trivial solution*) to the matrix equation $A\mathbf{x} = \mathbf{0}$. $\square$