Problem 1. Given \( A = \begin{pmatrix} 4 & 3 & -5 \\ -2 & -2 & 4 \\ -2 & -3 & 7 \end{pmatrix} \), observe that the first column minus three times the second column equals the third column. Find a nontrivial solution of \( Ax = 0 \).

Solution. We are given that
\[
\begin{pmatrix} -4 \\ -2 \\ -2 \end{pmatrix} - 3 \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 7 \end{pmatrix}.
\]
Thus,
\[
A \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ -2 \end{pmatrix} - 3 \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} - \begin{pmatrix} -5 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
\]
showing that \( x = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \) is nontrivial solution to \( Ax = 0 \).

Problem 2. True or False: If \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a linear transformation, then \( T(0) = 0 \), where the first 0 is in \( \mathbb{R}^n \) and the second 0 is in \( \mathbb{R}^m \). (If true, say why. If false, provide a counterexample.)

Solution. True. Since \( T \) is linear and 0 times any vector is 0,
\[
T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0.
\]
Problem 3. Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be the linear transformation given by
\[
T(x_1, x_2, x_3) = (x_1 + 2x_2 - 3x_3, 2x_1 + x_2 - 3x_3, -x_1 + x_2).
\]
(a) Determine the standard matrix for the linear transformation \( T \).
(b) Is \( T \) one-to-one? (Hint: Consider \( T(\mathbf{x}) = \mathbf{0} \).)

Solution. (a) Thinking of the input and output of \( T \) as column vectors, we have
\[
A = \begin{pmatrix}
T(e_1) & T(e_2) & T(e_3)
\end{pmatrix} = \begin{pmatrix}
1 & 2 & -3 \\
2 & 1 & -3 \\
-1 & 1 & 0
\end{pmatrix}.
\]

(b) Thus, the homogeneous matrix equation \( A\mathbf{x} = \mathbf{0} \) has infinitely many solutions. Since \( T\mathbf{x} = A\mathbf{x} \) for all \( \mathbf{x} \in \mathbb{R}^3 \), this shows that \( T \) maps infinitely many (and in particular, more than one) things to \( \mathbf{0} \), and cannot be one-to-one. \qed