February 13, 2014
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MATH 2940 (Dis. 204, 208, 216) – Quiz #2

Answer the following questions in the space provided. (There is a question on the back!)
Justify all answers.

Problem 1. Given $A = \begin{pmatrix} 4 & 3 & -5 \\ -2 & -2 & 4 \\ -2 & -3 & 7 \end{pmatrix}$, observe that the first column minus three times the second column equals the third column. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

Solution. We are given that

$$\begin{pmatrix} -4 \\ -2 \\ -2 \end{pmatrix} - 3 \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 7 \end{pmatrix}.$$

Thus,

$$A \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ -2 \end{pmatrix} - 3 \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} - \begin{pmatrix} -5 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

showing that $\mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$ is nontrivial solution to $A\mathbf{x} = \mathbf{0}$. □

Problem 2. True or False: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$, where the first $\mathbf{0}$ is in $\mathbb{R}^n$ and the second $\mathbf{0}$ is in $\mathbb{R}^m$. (If true, say why. If false, provide a counterexample.)

Solution. True. Since T is linear and 0 times any vector is $\mathbf{0}$,

$$T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0 \cdot T(\mathbf{0}) = \mathbf{0}.$$

□

Problem 3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 - 3x_3,\ 2x_1 + x_2 - 3x_3,\ -x_1 + x_2).$$

(a) Determine the standard matrix for the linear transformation T.

(b) Is T one-to-one? (Hint: Consider $T(\mathbf{x}) = \mathbf{0}$.)

Solution. (a) Thinking of the input and output of T as column vectors, we have

$$A = \begin{pmatrix} T(\mathbf{e_1}) & T(\mathbf{e_2}) & T(\mathbf{e_3}) \end{pmatrix} = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{pmatrix} \xrightarrow[R_e+R_1]{R_2-2R_1} \begin{pmatrix} 1 & 2 & -3 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \xrightarrow{R_3+R_2} \begin{pmatrix} 1 & 2 & -3 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, the homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. Since $T\mathbf{x} = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$, this shows that T maps infinitely many (and in particular, more than one) things to $\mathbf{0}$, and cannot be one-to-one. $\square$