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# MATH 2940 (Dis. 204, 208, 216) – Quiz #4

Answer the following questions in the space provided. (There is a question on the back!)
*Justify all answers.*

**Problem 1.** Below, $B$ is an echelon form of the matrix $A$.

$$A = \begin{pmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Find a basis for the column space of $A$, $\text{Col}(A)$.

(b) Find a basis for the null space of $A$, $\text{Nul}(A)$.

(c) Find numbers $R$, $M$, $K$ and $N$ so that:
$\text{Col}(A)$ is a $R$-dimensional subspace of $\mathbb{R}^M$ and $\text{Nul}(A)$ is a $K$-dimensional subspace of $\mathbb{R}^N$.

**Solution.** (a) Recall that a basis for $\text{Col}(A)$ is given by the pivot columns of $A$. Thus, a basis for $\text{Col}(A)$ is given by $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 8 \\ 9 \\ 9 \end{pmatrix} \right\}$.

(b) In order to find a basis for $\text{Nul}(A)$ we must solve $A\mathbf{x} = \mathbf{0}$ in parametric form. Using the echelon form $B$, we see that

$$\begin{aligned} x_5 &= 0 \\ x_4 &\text{ is free} \\ x_3 &= \frac{1}{3}(6x_4 - 3x_5) = 2x_4 \\ x_2 &\text{ is free} \\ x_1 &= -2x_2 - 2x_4 - 5x_5 = -2x_2 - 2x_4 \end{aligned}$$

So, $\mathbf{x} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, and thus $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for $\text{Nul}(A)$.

(c) $R = \dim \text{Col}(A) = 3$, $M = 4$, $K = \dim \text{Nul}(A) = 2$ and $N = 5$. □

**Problem 2.** The set $\mathcal{B} = \{1 - t^2, t - t^2, 2 - t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 1 + 3t - 6t^2$ relative to $\mathcal{B}$.

**Solution.**

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & -1 & 1 & -6 \end{array}\right) \xrightarrow{R_3 \to R_3 + R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & -1 & 3 & -5 \end{array}\right) \xrightarrow{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 2 & -2 \end{array}\right)$$

$$\xrightarrow[R_2 \to R_2 + \frac{1}{2} R_3]{R_1 \to R_1 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & -2 \end{array}\right) \xrightarrow{R_3 \to \frac{1}{2} R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Thus, $\mathbf{p}(t) = 3(1 - t^2) + 2(t - t^2) - (2 - t + t^2)$, and so the coordinate vector of $\mathbf{p}(t)$ relative to $\mathcal{B}$ is

$$[\mathbf{p}(t)]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

□