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MATH 2940 (Dis. 204, 208, 216) – Quiz #6

Answer the following questions in the space provided. (There is only one question, feel free to use the back for more space.) *Justify all answers.*

Problem 1. Consider the matrix

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 0 \end{pmatrix}$$

(a) Find the characteristic polynomial of A, and determine the eigenvalues of A.
(b) Find bases for each of the eigenspaces of A.
(c) Find an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.
(d) Find an invertible matrix P' and diagonal matrix D' such that $A^{-1} = P'D'P'^{-1}$. *Hint*: You've done most of the work already in part (c).

(You do *not* need to verify that $A = PDP^{-1}$, or $AP = PD$, etc, in parts (c) and (d).)

Solution. (a)

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & -1 & -\lambda \end{vmatrix} = (-1)^{2+2}(2-\lambda)\begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 1).$$

Thus, the characteristic polynomial of A is $p(\lambda) = (2-\lambda)(\lambda^2-1)$, and the eigenvalues of A are $-1, 1, 2$.

(b)

$$A - (-1)I = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ so the eigenspace consists of vectors } \mathbf{x} = x_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

with basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$$A - I = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ so the eigenspace consists of vectors } \mathbf{x} = x_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

with basis $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$$A - 2I = \begin{pmatrix} -2 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$ so the eigenspace consists of vectors $\mathbf{x} = x_3 \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$,

with basis $\left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right\}$.

(c) Let $P = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & -3 \\ 1 & 1 & 1 \end{pmatrix}$, and $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. (Answers given by permuting the columns of P and the eigenvalues of D in the corresponding way will also work.)

(d) Recall that if $A = PDP^{-1}$, and A is invertible, then $A^{-1} = PD^{-1}P^{-1}$. Thus, taking $P' = P = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & -3 \\ 1 & 1 & 1 \end{pmatrix}$, and $D' = D^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$ will work.