Classifying Classification

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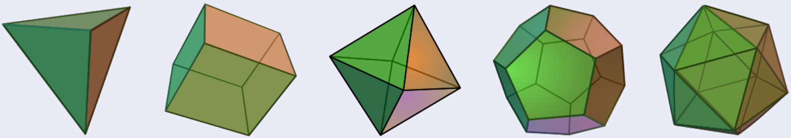
Classification in mathematics

Given some class of mathematical objects, it is natural to ask if we can "classify" all such objects, up to some notion of equivalence.

Examples are sprinkled throughout the history of mathematics:

Theorem (Euclid (?), c. 300 BC)

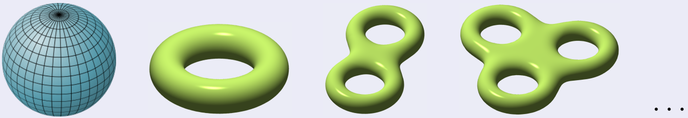
There are exactly five regular convex polyhedra (the Platonic solids), up to similarity:



Classification in mathematics (cont'd)

Theorem

Every connected orientable closed surface is a sphere, or a connected sum of $g \geq 1$ tori, up to homeomorphism:



Theorem

Every finitely generated abelian group is of the form

$$\mathbb{Z}^k \times (\mathbb{Z}/p_1\mathbb{Z})^{k_1} \times \cdots \times (\mathbb{Z}/p_n\mathbb{Z})^{k_n},$$

for some unique set of primes $p_1, \ldots, p_n$ and non-negative integers $k, k_1, \ldots, k_n$, up to isomorphism.

Classification and equivalence relations

In all of these examples, the notion of equivalence (similarity of shapes, homeomorphism of spaces, isomorphism of groups) is an equivalence relation on the class $\mathcal{C}$ of objects we are considering. That is, if $x\,E\,y$ denotes that x is equivalent to y, then for all x, y and z in $\mathcal{C}$:

- $x\,E\,x$.
- If $x\,E\,y$, then $y\,E\,x$.
- If $x\,E\,y$ and $y\,E\,z$, then $x\,E\,z$.

Invariants and reduction

In many examples, to each object x in our class $\mathcal{C}$, we can associate another object, often a number or a string of numbers, say n_x, such that

$$x \, E \, y \quad \text{implies} \quad n_x = n_y.$$

So, the association $x \mapsto n_x$ produces an invariant, up to E-equivalence.

If, moreover, we have that

$$x \, E \, y \quad \text{if and only if} \quad n_x = n_y,$$

then we say that the association $x \mapsto n_x$ is a reduction. In this case, we have reduced the problem of checking if objects in our class are E-equivalent to computing the above invariants, and comparing them.

Invariants and reduction (cont'd)

Example

The association of a Platonic solid to its number of faces is a reduction.

Example

The association of a connected orientable closed surface to its genus (i.e., the number of "holes" it has) is a reduction.

Example

The association of a finitely generated abelian group A to the string $(k, p_1, \ldots, p_2, \ldots \ldots, p_n \ldots) \in \mathbb{N}^{<\infty}$, where $p_1 < p_2 < \cdots < p_n$ are primes listed with multiplicity $k_1, k_2, \ldots, k_n$, and

$$A \cong \mathbb{Z}^k \times (\mathbb{Z}/p_1\mathbb{Z})^{k_1} \times \cdots \times (\mathbb{Z}/p_n\mathbb{Z})^{k_n},$$

is a reduction.

A general framework for classification?

We wish to develop a general framework for studying classification problems. We will restrict ourselves to considering only countable, or countable-like (i.e., separable) objects, e.g., countable groups, separable metric spaces, etc.

- First, we need to “code” our class of objects into a space, in a reasonable way.
- Then, our notion of equivalence on the objects induces an equivalence relation on the space.
- We can generalize the idea of reduction to enable us to compare different classification problems, produce invariants, prove “hardness” results, etc.

A simple example

Suppose we are interested in classifying all structures of the form $\langle C, U\rangle$ where C is a countably infinite set, and U is a subset of C, also called a unary relation or unary predicate on C.

Example

If $C = \mathbb{N}$, and $U = \{3, 5, 6\}$, then $\langle C, U\rangle$ looks like:

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \ldots$$

Example

If $C = \mathbb{N}$, and $U = \{n : n \text{ is odd}\}$, then $\langle C, U\rangle$ looks like:

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \ldots$$

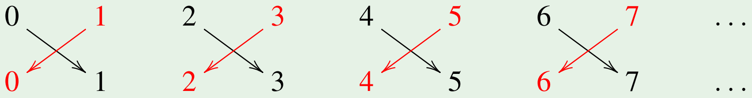
A simple example (cont'd)

If $\langle C, U \rangle$ and $\langle D, V \rangle$ are two such structures, then a map $f : C \to D$ is an isomorphism if f is a bijection, and for all $x \in C$, we have $x \in U$ if and only if $f(x) \in V$.

Example

Consider the structures $\langle \mathbb{N}, O \rangle$, where $O = \{n : n \text{ is odd}\}$, and $\langle \mathbb{N}, E \rangle$, where $E = \{n : n \text{ is even}\}$. The function $f : \mathbb{N} \to \mathbb{N}$ given by

$$f(n) = \begin{cases} n - 1 & \text{if } n \text{ is odd,} \\ n + 1 & \text{if } n \text{ is even,} \end{cases}$$



is an isomorphism of $\langle \mathbb{N}, O \rangle$ onto $\langle \mathbb{N}, E \rangle$.

A simple example (cont'd)

There are many (too many) different countably infinite sets, but there is no apparent harm in fixing the underlying countable set, so we choose our favorite one: $\mathbb{N}$.

Every structure $\langle \mathbb{N}, U \rangle$ can be uniquely identified as a point in the space of countably infinite binary sequences $\{0,1\}^{\mathbb{N}}$ as follows:

$$\langle \mathbb{N}, U \rangle \text{ corresponds to } x = (x_n)_{n \in \mathbb{N}}, \text{ where } x_n = \begin{cases} 1 & \text{if } n \in U, \\ 0 & \text{if } n \notin U. \end{cases}$$

Example

The structure $\langle \mathbb{N}, O \rangle$ where $O = \{n : n \text{ is odd}\}$ corresponds to the sequence $(0,1,0,1,0,1,0,1,\ldots) \in \{0,1\}^{\mathbb{N}}$.

A simple example (cont'd)

When are two structures $\langle C, U\rangle$ and $\langle D, V\rangle$ isomorphic?

Exactly when $|U| = |V|$ *and* $|C \setminus U| = |D \setminus V|$.

Thus, the association of a structure $\langle C, U\rangle$ to the pair (m, n) where $m = |U|$ and $n = |C \setminus U|$, one of which may be ∞, yields a reduction of isomorphism on our class of structures to pairs in $\mathbb{N} \cup \{\infty\}$.

From the coding of these structures into the space $\{0, 1\}^{\mathbb{N}}$, we have an equivalence relation on $\{0, 1\}^{\mathbb{N}}$ given by $x\,E\,y$ if and only if the corresponding structures are isomorphic.

The reduction above induces a function $f : \{0, 1\}^{\mathbb{N}} \to (\mathbb{N} \cup \{\infty\})^2$, given by $f(x) = (m, n)$, where x corresponds to the structure $\langle \mathbb{N}, U\rangle$, $m = |U|$ and $n = |\mathbb{N} \setminus U|$. This satisfies $x\,E\,y$ if and only if $f(x) = f(y)$.

We will return to this example later.

Coding structures into spaces

This method of coding countable structures into spaces generalizes: If we wish to consider structures of the form $\langle C, (R_i)_{i \in I} \rangle$, where C is a countably infinite set, I is countable, and each R_i is an n_i-ary relation on C (i.e., $R_i \subseteq C^{n_i}$), by taking $C = \mathbb{N}$, we can identify this class of structures with the space

$$\prod_{i \in I} \{0, 1\}^{\mathbb{N}^{n_i}}$$

in similar way.

By considering n-ary functions as $(n+1)$-ary relations (via their graphs), we can realize the classes of all countably infinite groups, rings, linear orders, partial orders, ordered groups, etc, as subspaces of a space constructed as above. Equivalence relations on the class of structures, such as isomorphism, induce equivalence relations on the corresponding space.

Coding structures into spaces (cont'd)

If we are interested in more general *separable* structures, these can also be coded. For example, every separable complete metric space is isometric to a closed subspace of the *Urysohn space* $\mathbb{U}$, so we can identify the class of separable complete metric spaces with the space of closed subsets of $\mathbb{U}$, itself a separable complete metric space with an appropriate metric.

In all of these cases, the spaces constructed belong to a well-studied class, on which the study of "nice" equivalence relations provides a general framework.

Polish spaces

Definition

A Polish space is a separable, completely metrizable topological space. i.e., a separable space whose topology can be induced by some complete metric.

Examples

Countable discrete spaces, $[0, 1]$, $\mathbb{R}$, $\ell^2(\mathbb{N})$, $\{0, 1\}^{\mathbb{N}}$, ...

Fact

Countable products, countable disjoint unions, and closed subspaces of Polish spaces are Polish.

Borel and analytic sets

We want to focus our attention on “definable” subsets:

Definition

A subset of a Polish space X is Borel if it is contained in the smallest σ-algebra generated by the open sets. i.e., it is obtained from open sets by taking countable unions and complements.

Definition

A subset A of a Polish space X is analytic if it is a projection of a Borel subset $B \subseteq X \times Y$, where Y is also Polish, i.e.,

$$A = \pi_X(B) = \{x \in X : \exists y \in Y((x, y) \in B)\}.$$

Borel and analytic equivalence relations

Definition

An equivalence relation $E = \{(x,y) \in X \times X : x \, E \, y\}$ on a Polish space X is Borel (analytic) if it is Borel (analytic) as a subset of $X \times X$.

Example

Let X be an Polish space. Equality on X, represented by

$$\Delta(X) = \{(x,y) \in X \times X : x = y\}$$

is a Borel equivalence relation on X (in fact, it is closed).

Borel and analytic equivalence relations (cont'd)

Example

Let $X = \mathbb{R}$. Consider the equivalence relation E_V given by

$$x \; E_V \; y \quad \text{if and only if} \quad x - y \in \mathbb{Q}.$$

E_V is a Borel equivalence relation on $\mathbb{R}$, called Vitali equivalence.

Example

Let $X = \{0,1\}^{\mathbb{N}}$. Consider the equivalence relation E_0 given by

$$(x_n)_{n \in \mathbb{N}} \; E_0 \; (y_n)_{n \in \mathbb{N}} \quad \text{if and only if} \quad \exists n \forall m \geq n (x_m = y_m),$$

i.e., eventual equivalence of infinite binary sequences, is a Borel equivalence relation on $\{0,1\}^{\mathbb{N}}$.

Orbit equivalence

A Polish group G is a group and a Polish space, such that the group operations are continuous. e.g., all countable discrete groups, Lie groups, separable locally compact groups, etc.

Example

Let G be a Polish group which acts continuously on a Polish space X, i.e., the map $G \times X \to X : (g, x) \mapsto g \cdot x$ is continuous. Then, the orbit equivalence relation on X induced by this action,

$$E_G = \{(x, y) \in X \times X : \exists g \in G(g \cdot x = y)\},$$

is an analytic equivalence relation on X. If G is countable, then E_G is a Borel equivalence relation on X.

Orbit equivalence (cont'd)

Many interesting examples of equivalence relations arise as orbit equivalence relations of Polish group actions:

Example

Identifying $\{0,1\}^{\mathbb{N}}$ with the product group $\prod_{n\in\mathbb{N}}(\mathbb{Z}/2\mathbb{Z})$, then the orbit equivalence relation induced by the left multiplication action of the subgroup $\oplus_{n\in\mathbb{N}}(\mathbb{Z}/2\mathbb{Z})$ on $\prod_{n\in\mathbb{N}}(\mathbb{Z}/2\mathbb{Z})$ is exactly E_0.

Orbit equivalence (cont'd)

Example

Recall our coding of structures $\langle \mathbb{N}, U \rangle$ as the space $\{0,1\}^{\mathbb{N}}$. Let E denote the equivalence on $\{0,1\}^{\mathbb{N}}$ relation induced by isomorphism on the structures. Let S_∞, the group of all permutations of $\mathbb{N}$, act on $\{0,1\}^{\mathbb{N}}$ by

$$g \cdot (x_n)_{n\in\mathbb{N}} = (y_n)_{n\in\mathbb{N}} \quad \text{where} \quad y_n = x_{g^{-1}(n)}.$$

This action is continuous and has orbit equivalence relation E: If $(x_n)_{n\in\mathbb{N}}$ corresponds to the structure $\langle \mathbb{N}, U \rangle$, then $g \cdot (x_n)_{n\in\mathbb{N}}$ corresponds to the structure $\langle \mathbb{N}, g(U) \rangle$.

There is an analogous action of S_∞ on the space used to code all structures of the form $\langle \mathbb{N}, (R_i)_{i\in I} \rangle$, and again, the orbit equivalence is exactly the equivalence relation induced by isomorphism. This is called the logic action.

Borel functions

Much like with sets, the following gives a notion of “definable” functions between Polish spaces:

Definition

A function $f : X \to Y$ between Polish spaces X and Y is Borel if its graph $\{(x, y) \in X \times Y : f(x) = y\}$ is Borel as a subset of $X \times Y$. Equivalently, for every Borel set $B \subseteq Y$, $f^{-1}(B)$ is Borel in X.

Borel reduction

We now have the machinery to compare equivalence relations:

Definition

If E and F are equivalence relations on Polish spaces X and Y, respectively, then a map $f : X \to Y$ is a Borel reduction of E to F if f is Borel and

$$x \, E \, y \quad \text{if and only if} \quad f(x) \, F \, f(y).$$

In this case, we say E is Borel reducible to F, and write $E \leq_B F$.

This is a "definable" generalization of our original notion of reduction: a Borel reduction of E to F reduces the problem of checking if $x \, E \, y$ to "computing" with f, and checking if $f(x) \, F \, f(y)$.

Borel equivalence

Definition

If E and F are equivalence relations on Polish spaces X and Y, respectively, we say that E and F are Borel bireducible, written $E \sim_B F$, if $E \leq_B F$ and $F \leq_B E$. If $E \leq_B F$ but $E \not\sim_B F$, then we write $E <_B F$.

Example

For Polish spaces X and Y, $\Delta(X) \sim_B \Delta(Y)$ if and only if $|X| = |Y|$.

Example

If E_V denotes Vitali equivalence on $\mathbb{R}$, and E_0 is as before, then $E_V \sim_B E_0$.

Smooth equivalence relations

Definition

A Borel equivalence relation E on a Polish space X is smooth (or concretely classifiable) if it is Borel reducible to $\Delta(Y)$, for some Polish space Y.

The smooth equivalence relations are exactly those which admit reductions to *real-valued invariants*. Modulo Borel bireducibility, they are, ordered by $\leq_B$:

$$\Delta(\{0\}) <_B \Delta(\{0,1\}) <_B \Delta(\{0,1,2\}) <_B \cdots <_B \Delta(\mathbb{N}) <_B \Delta(\mathbb{R}).$$

Non-smooth equivalence relations

Theorem

*The Borel equivalence relation E_0 on $\{0,1\}^{\mathbb{N}}$ is **not** smooth.*

Proof (sketch).

One can show that if E_0 is smooth, then there is a Borel set $B \subseteq \{0,1\}^{\mathbb{N}}$ such that B meets every E_0-equivalence class in exactly one point. Identify $\{0,1\}^{\mathbb{N}}$ with the group $\prod_{n\in\mathbb{N}}(\mathbb{Z}/2\mathbb{Z})$, and let the subgroup $\Gamma = \oplus_{n\in\mathbb{N}}(\mathbb{Z}/2\mathbb{Z})$ act by left multiplication. If $g, g' \in \Gamma$, $g \neq g'$, then $g \cdot B$ and $g' \cdot B$ are disjoint Borel sets, and $\{0,1\}^{\mathbb{N}} = \bigcup_{g\in\Gamma} g \cdot B$. Let μ be the product (Haar) measure on $\{0,1\}^{\mathbb{N}}$, which is Γ-invariant. Then,

$$1 = \mu(\{0,1\}^{\mathbb{N}}) = \mu(\bigcup_{g\in\Gamma} g \cdot B) = \sum_{g\in\Gamma} \mu(g \cdot B) = \sum_{g\in\Gamma} \mu(B),$$

which is absurd, since we cannot have $\mu(B) = 0$ or $\mu(B) > 0$. □

Degree structure of Borel equivalence relations

What does the partial order $\leq_B$ (modulo Borel bireducibility) look like? Work of Silver (1980) and Harrington-Kechris-Louveau (1990) shows the following is an initial segment of the Borel equivalence relations:

$$\Delta(\{0\}) <_B \Delta(\{0,1\}) <_B \Delta(\{0,1,2\}) <_B \cdots <_B \Delta(\mathbb{N}) <_B \Delta(\mathbb{R}) <_B E_0$$

However, $\leq_B$ is **not** a linear order. In fact, Adams-Kechris (2000) showed that *every* Borel partial ordering on any Polish space can be embedded into $\leq_B$, even when restricted to those Borel equivalence relations whose equivalence classes are all countable.

The land beyond E_0 may best be described as “messy”.

Back to classification

Returning to our original motivation, suppose that $\mathcal{C}$ is a collection of structures that we wish to classify up to some notion of equivalence $\mathcal{E}$. If we have coded $\mathcal{C}$ as a Polish space X, then $\mathcal{E}$ induces an equivalence relation E on X. In many examples, E is Borel, or at least analytic.

Borel reduction gives us a way of measuring the difficulty of E. For example, if E is smooth, the classification problem is *not too hard*.

However, if $F \leq_B E$, and we already know that F is hard, then we know that E is *at least as hard* as F.

A caveat

Warning: This scheme we've developed for understanding classification problems largely ignores the difficulties in classifying finite objects.

For example, the class of all finite groups can be easily coded as a countably infinite discrete space, and thus isomorphism of finite groups is Borel reducible to $\Delta(\mathbb{N})$, which means that $\leq_B$ thinks that classifying all finite groups up to isomorphism is "easy".

Experience, and the 1000+ pages it took to classify all finite *simple* groups, beg to differ.

Back to our example

We have coded all countably infinite structures of the form $\langle C, U \rangle$ into the Polish space $\{0,1\}^{\mathbb{N}}$, and the isomorphism relation on the structures induces an equivalence relation E on $\{0,1\}^{\mathbb{N}}$. It is not hard to check that this equivalence relation is Borel.

We defined a function $f : \{0,1\}^{\mathbb{N}} \to (\mathbb{N} \cup \{\infty\})^2$ by $f(x) = (m,n)$, where if x corresponds to the structure $\langle \mathbb{N}, U \rangle$, then $m = |U|$ and $n = |\mathbb{N} \setminus U|$. This satisfies $x \, E \, y$ if and only if $f(x) = f(y)$.

It is not hard to check that f is Borel, and hence gives a Borel reduction of E to $\Delta((\mathbb{N} \cup \{\infty\})^2) \sim_B \Delta(\mathbb{N})$. Thus, E is smooth, and in fact, we can definably assign natural number invariants.

Examples from group theory

Theorem (Thomas-Velickovic, 1999)

If $\cong_{fg}$ denotes isomorphism of finitely generated groups, then $E_0 <_B \cong_{fg}$. In fact, $E \leq_B \cong_{fg}$ whenever E is a Borel equivalence relation with countable classes ($\cong_{fg}$ is "complete" for this class).

Theorem (Baer, 1937)

Isomorphism of torsion-free abelian groups of rank one is Borel bireducible to E_0, via the "root-type" of its elements.

Theorem (Thomas, 2003)

If $\cong_n$ denotes the isomorphism relation for torsion-free abelian groups of rank n, then

$$\cong_1 <_B \cong_2 <_B \cdots <_B \cong_n <_B \cdots .$$

Other examples

Theorem (Hjorth-Kechris, 2000)

Conformal equivalence of Riemann surfaces is complete for Borel equivalence relations with countable classes. The two-dimensional analog is "much" more complicated.

Theorem (Ferenczi-Louveau-Rosendal, 2009)

*The following are complete analytic equivalence relations, i.e., **all** analytic equivalence relations are Borel reducible to them:*

1. *Uniform homeomorphism of complete separable metric spaces.*
2. *Topological isomorphism of Polish groups.*
3. *Isomorphism of separable Banach spaces.*

Further directions

Where do we go from here?

There are many open questions involving well-known equivalence relations on classes of structures. From group theory:

Question

What is the Borel complexity of the quasi-isometry and virtual isomorphism relations on finitely generated groups?

From computability theory:

Question

Is Turing equivalence on $\{0, 1\}^{\mathbb{N}}$ *complete for the Borel equivalence relations with countable classes?*

Further directions (cont'd)

Since many equivalence relations come from group actions, it's natural to ask whether "nice" groups produce "nice" equivalence relations. For example, very recent work (Gao-Jackson, to appear) has shown that if E is induced by a countable abelian group, then $E \leq_B E_0$. The following is a long-standing open question:

Question

If E is induced by a Borel action of a countable amenable group, then is $E \leq_B E_0$?

Thanks!