A Crash Course in Topological Groups

Iian B. Smythe

Department of Mathematics
Cornell University

Olivetti Club
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Outline

1. Examples and definitions

2. Basic topological properties

3. Basic "algebraic" properties

4. Further Results
   - Harmonic analysis on locally compact groups
   - Descriptive set theory on Polish groups
   - Hilbert's Fifth Problem

Motivating examples

- A Lie group *G* is a group, which is also a smooth manifold, such that the group operations (multiplication and inversion) are smooth. In particular, *G* is a topological space such that the group operations are continuous.

- A Banach space *X* is a complete normed vector space. In particular, *X* is an abelian group and a topological space such that the group operations (addition and subtraction) are continuous.

Main Definition

Definition

A *topological group* is a group G, which is also a topological space, such that the group operations,

- $\times : G \times G \to G$, where $G \times G$ has the product topology, and
- $^{-1} : G \to G$,

are continuous.

Examples

- Lie groups: $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O(n)$, $U(n)$, $PSL_2(\mathbb{C})$ ...
- The underlying additive group of a Banach space (or more generally, topological vector space): $\mathbb{R}^n$, $L^p(X, \mu)$, C_0, $C_c(X)$, ...
- Groups of homeomorphisms of a 'nice' topological space, or diffeomorphisms of a smooth manifold, can be made into topological groups.
- Any group taken with the discrete topology.
- Any (arbitrary) direct product of these with the product topology. Note that, for example, $(\mathbb{Z}/2\mathbb{Z})^\omega$ is *not* discrete with the product topology (it is homeomorphic to the Cantor set).

Homogeneity

- A topological group acts on itself by certain canonical self-homeomorphisms: inversion, left (or right) translation by a fixed element, and conjugation by a fixed element.
- Translation by elements gives a topological group a *homogeneous* structure, i.e. we can move from a point h in the group to any other point k by the homeomorphism $g \mapsto kh^{-1}g$.
- This allows us to infer certain topological information about the whole group from information at any particular point (such as the identity).

Homogeneity (cont'd)

Proposition

Let G and H be topological groups.

- *Denote by $\mathcal{N}_G$ the set of all open neighbourhoods of e in G. Then, $\{gU : g \in G, U \in \mathcal{N}_G\}$ is exactly the topology on G.*
- *A group homomorphism $\varphi : G \to H$ is continuous on G if and only if φ is continuous at one point of G.*

Proof.

If V is any nonempty open set in G, it contains some $g \in G$.
$g^{-1}V \in \mathcal{N}_G$ and $V = g(g^{-1}V)$. This proves the first claim.
We may assume that φ is continuous at e_G. Let $g \in G$, and $W \subseteq H$ an open set containing $\varphi(g)$. By continuity at e_G, there is $U \in \mathcal{N}_G$ with $\varphi(U) \subseteq \varphi(g)^{-1}W$. So, $\varphi(gU) = \varphi(g)\varphi(U) \subseteq W$. □

Homogeneity (cont'd)

Similarly, we can extend many local properties from any particular point in a topological group to the whole group.

Proposition

Let G be a topological group.

- *G is locally compact if and only if there is one point of G with a local basis of compact sets.*
- *G is locally (path) connected if and only if there is one point of G with a local basis of open, (path) connected sets.*
- *G is locally euclidean if and only if there is one point of G with a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$.*
- *G is discrete if and only if there is one point of G which is isolated.*

Separation Properties

Recall the following terminology:

Definition

Let X be a topological space.

- X is T_0 if for any pair of distinct points $x, y \in X$, there exists an open set containing one but not the other.
- X is T_1 if for any pair of distinct points $x, y \in X$, there are open sets containing each point, and not containing the other.
- X is T_2 (*Hausdorff*) if for any pair of distinct points $x, y \in X$, there are disjoint open sets containing each.
- X is T_3 if X is T_1 and *regular*: i.e. if $U \subseteq X$ is an open set containing x, then there is an open set V with $x \in V \subseteq \overline{V} \subseteq U$.

In general, $T_3 \implies T_2 \implies T_1 \implies T_0$, and these implications are strict.

Separation Properties (cont'd)

The distinctions between the aforementioned separation properties dissolve in a topological group.

Lemma

Let G be a topological group. For every $U \in \mathcal{N}_G$, we have $\overline{U} \subseteq U^{-1}U$.

Proof.

Let $g \in \overline{U}$. Since Ug is an open neighbourhood of g, $Ug \cap U \neq \emptyset$. That is, there are $u_1, u_2 \in U$ with $u_1 g = u_2$, so $g = u_1^{-1} u_2 \in U^{-1}U$. $\qquad\square$

Separation Properties (cont'd)

Proposition

Every topological group G is regular, and if G is T_0, then G is T_3.

Proof.

Let $U \in \mathcal{N}_G$. By continuity of multiplication and inversion, one can show that there is $V \in \mathcal{N}_G$ such that $VV \subseteq U$ and $V^{-1} = V$. By the lemma, $\overline{V} \subseteq V^{-1}V = VV \subseteq U$. Hence, G is regular at e, which suffices by homogeneity.

Suppose G is T_0, and let $g, h \in G$ be distinct. We may assume that there is $U \in \mathcal{N}_G$ such that $h \notin Ug$. So, $hg^{-1} \notin U$, and thus, $g \notin U^{-1}h$. That is, G is T_1 and regular, hence T_3. $\qquad\square$

Separation Properties (cont'd)

In fact, more is true (with significantly more work):

Theorem

Every topological group G is completely regular, i.e. for any closed $F \subseteq G$ with $g \notin F$, there is a continuous function $f : G \to [0, 1]$ such that $f(g) = 0$ and $f(F) = \{1\}$. If G is T_0, then G is $T_{3\frac{1}{2}}$ (or Tychonoff).

Further, a result of Birkhoff and Kakutani shows that topological groups satisfy the "ultimate" separation property, metrizability, under very weak assumptions:

Theorem

A topological group G is metrizable if and only if G is T_0 and first countable (i.e. every point has a countable local basis).

"Algebra" with topological groups

- The class of all topological groups forms a category where the morphisms are *continuous* group homomorphisms.
- Isomorphisms in this setting are continuous group isomorphisms with *continuous inverses*, and embeddings are continuous group embeddings which are *open onto their images*.
- Some of the classical constructions from group theory can be carried over to topological groups in a natural way.

"Algebra" with topological groups (cont'd)

Definition

Let G be a topological group, and N a normal subgroup. The *quotient topological group* of G by N is the group G/N together with the topology formed by declaring $U \subseteq G/N$ open if and only if $\pi^{-1}(U)$ is open in G, where $\pi : G \to G/N$ is the canonical projection.

$\pi : G \to G/N$ is a *quotient map* in the topological sense, i.e. it is continuous, open and surjective. However, care must be taken when forming quotients in light of the following (easy) fact:

Proposition

G/N is Hausdorff if and only if N is a closed normal subgroup of G.

"Algebra" with topological groups (cont'd)

The familiar UMP for quotients holds in this setting:

Proposition

Let G and H be topological groups, $\varphi : G \to H$ a continuous group homomorphism, and N a normal subgroup of G. If $N \subseteq \ker(\varphi)$, then there exists a unique continuous group homomorphism $\tilde{\varphi} : G/N \to H$ such that $\varphi = \tilde{\varphi}\pi$. That is,

$$
\begin{array}{ccc}
G & \xrightarrow{\ \varphi\ } & H \\
{\scriptstyle \pi}\big\downarrow & \nearrow & \\
G/N & {\scriptstyle \tilde{\varphi}} &
\end{array}
$$

"Algebra" with topological groups (cont'd)

What about the isomorphism theorems?

Proposition

The First Isomorphism Theorem fails for topological groups.

Proof.

Consider the additive group $\mathbb{R}$ with its usual topology, and denote by $\mathbb{R}_d$ the additive group of reals with the discrete topology. The identity map $\text{id} : \mathbb{R}_d \to \mathbb{R}$ is a continuous group homomorphism with trivial kernel, but $\mathbb{R}_d / \ker(\text{id}) = \mathbb{R}_d \ncong \mathbb{R}$ as *topological* groups. $\qquad\square$

"Algebra" with topological groups (cont'd)

Proposition

The Second Isomorphism Theorem fails for topological groups.

Proof.

Consider the additive group $\mathbb{R}$ with its usual topology, and the closed (normal) subgroups $N = \mathbb{Z}$ and $H = \{\alpha n : n \in \mathbb{Z}\}$, where α is irrational. Clearly $H \cap N = \{0\}$, and so $H \cong H/(H \cap N)$ as topological groups; in particular, both are discrete. However, $H + N$ is a dense subset of $\mathbb{R}$, so $(H + N)/N$ is a dense subset of the compact group $\mathbb{R}/\mathbb{Z} \cong U(1)$. In particular, $(H + N)/N$ is not discrete, and $(H + N)/N \not\cong H/(H \cap N)$ as topological groups. $\qquad \square$

Strangely enough, the Third Isomorphism Theorem holds (exercise).

The Connected Component of the Identity

An important, and elementary, example of the interplay between algebra and topology within a topological group is the connected component of the identity. Recall from topology:

Definition

Let X be a topological space, and $x \in X$. The connected component of x in X is the largest connected subset of X containing x.

Proposition

Let G be a topological group, and denote by G_0 the connected component of e in G. Then, G_0 is a closed, normal subgroup of G.

The Connected Component of the Identity (cont'd)

Proof.

That G_0 is closed is topology (connected components are closed). We know that $e \in G_0$ by definition. Let $g, h \in G_0$. Multiplication on the right is a homeomorphism, so $G_0 h^{-1}$ is connected, and it contains e since $h \in G_0$. Thus, $G_0 h^{-1} \subseteq G_0$, and in particular $gh^{-1} \in G_0$, showing that G_0 is a subgroup.

Conjugation by any fixed element of G is a homeomorphism which sends $e \mapsto e$, so G_0 is normal in G (in fact, G_0 is fully invariant under *continuous* endomorphisms of G). $\square$

Corollary

Let G be a topological group, then G/G_0 is the (totally disconnected) group of connected components of G.

Further results

- We will now survey some deeper results on certain classes of topological groups relevant to analysts, set theorists, and topologists, respectively.
- For the remainder of this talk, all topological groups are assumed to be T_0, and in particular Hausdorff.

Harmonic analysis on locally compact groups

Definition

A topological group *G* is *locally compact* if the underlying space is locally compact. Equivalently, *e* has a compact neighbourhood.

The following theorem, due to Haar (1933) and Weil (1940), allows us to do analysis in a meaningful way on such groups:

Theorem

Every locally compact group G admits a (left) translation invariant Borel measure, called the (left) Haar measure μ on G. That is, μ is a measure on the Borel subsets of G such that

- $\mu(gB) = \mu(B)$, for all $g \in G$, and Borel sets $B \subseteq G$.

Moreover, $\mu(K)$ is finite for every compact $K \subseteq G$, and μ is unique up to multiplication by a positive constant.

Harmonic analysis on locally compact groups (cont'd)

The proof establishing the existence of the Haar measure is nonconstructive, but many examples can be given explicitly.

- Lebesgue measure on $\mathbb{R}^n$.
- $|\det(X)|^{-n}dX$ on $GL_n(\mathbb{R})$, where dX is Lebesgue measure on $\mathbb{R}^{n \times n}$.
- Counting measure on any discrete group.

Descriptive Set Theory and Polish Groups

The following important class of spaces is a core object of study in *descriptive set theory*:

Definition

A topological space *X* is said to be *Polish* if it is separable and completely metrizable.

Naturally,

Definition

A topological group *G* is said to be *Polish* if its underlying space is Polish.

Examples include Lie groups, separable Banach spaces, countable discrete groups, and any countable products of these.

Descriptive Set Theory and Polish Groups (cont'd)

Definition

Let X be a topological space.

- $A \subseteq X$ is *nowhere dense*, if the interior of the closure of A is empty.
- $M \subseteq X$ is *meager* if its contained in a countable union of nowhere dense sets
- $B \subseteq X$ has the *Baire property* if there is an open set $U \subseteq X$ such that $B \Delta U$ is meager.

Definition

Let X and Y be topological spaces. A function $f : X \to Y$ is *Baire measurable* if inverse images of open sets have the Baire property.

It can be shown that every Borel subset of a Polish space has the Baire property.

Descriptive Set Theory and Polish Groups (cont'd)

The following theorem of Pettis (1950) shows that, in some sense, every homomorphism that we can describe between Polish groups is continuous.

Theorem

Let G and H be Polish groups. If $\varphi : G \to H$ is a Baire measurable homomorphism, then φ is continuous.

Shelah (1984) showed that it is consistent with ZF that every subset of every Polish space has the Baire property. Hence, it is consistent with ZF that every homomorphism between Polish groups is continuous, and constructing non-continuous homomorphisms between such groups requires use of the Axiom of Choice.

Hilbert's Fifth Problem

In 1900, David Hilbert published a list of what he considered to be the 23 most important problems in mathematics.

The fifth problem on this list was stated as follows: *How far Lie's concept of continuous groups of transformations is approachable in our investigations without the assumption of the differentiability of the functions.*

This problem is notably vague and has been understood in many different ways. The interpretation which received the most attention was as follows: *Is every locally euclidean topological group isomorphic to a Lie group?*

Hilbert's Fifth Problem (cont'd)

von Neumann (1933) established the positive answer to the previous question in the case of compact groups, and work of Gleason (1952) and Montgomery-Zippin (1952) established the general answer:

Theorem

A topological group G is locally euclidean if and only if it is isomorphic to a Lie group.

Moreover, a locally euclidean group can be (uniquely) endowed with the structure of a real-analytic manifold.

Hilbert's Fifth Problem (cont'd)

Due to the vagueness of Hilbert's original question, the story of this problem is far from over. The following result, due to Goldbring (2010), concerns spaces having group operations locally around an identity:

Theorem

A local group G is locally euclidean if and only if a restriction of G to an open neighbourhood of the identity is isomorphic to a local Lie group.

Alternatively, one can also interpret Hilbert's problem in the form of the Hilbert-Smith Conjecture:

Conjecture

If G is a locally compact group which acts continuously and effectively on a connected manifold M, then G is isomorphic to a Lie group.

This question remains open to this day.