It's all fun and games...

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Informally, a *Gale-Stewart game* is an infinite, two player game of perfect information wherein the two players, say Alice and Bob, alternate playing natural numbers, building an infinite sequence of digits, i.e., a real number. The game comes with a specified target set, say A, and Alice wins if the real they construct is in A, while Bob wins otherwise. For technical reasons, we will find it convenient to replace the reals with elements of *Baire space*, $\mathbb{N}^{\mathbb{N}}$, the space of sequences of natural numbers, endowed with the product topology. [As a side note, this space is known to be homeomorphic (via continued fractions) to the space of irrational numbers with the subspace topology inherited from $\mathbb{R}$.]

Definition 1. Let $A \subseteq \mathbb{N}^{\mathbb{N}}$. The *Gale-Stewart game* corresponding to A, denoted by G_A, is defined as follows: a *play* of the game is a sequence $\langle a_0, b_0, a_1, b_1, \ldots\rangle \in \mathbb{N}^{\mathbb{N}}$, where for each n, a_n is the nth *move* of player I (Alice), and b_n is the nth move of player II (Bob):

n	0	1	2	$\cdots$
Alice	a_0	a_1	a_2	$\cdots$
Bob	b_0	b_1	b_2	$\cdots$

Alice wins a play $\langle a_0, b_0, a_1, b_1, \ldots\rangle$ of G_A if $\langle a_0, b_0, a_1, b_1, \ldots\rangle \in A$, and Bob wins the play otherwise.

A *strategy* for Alice is a function σ from finite sequences in $\mathbb{N}$ of even length to $\mathbb{N}$. Alice plays $\langle a_0, b_0, a_1, b_1, \ldots\rangle$ *by the strategy* σ if $\sigma(\emptyset) = a_0$, $\sigma(\langle a_0, b_0\rangle) = a_1$, $\sigma(\langle a_0, b_0, a_1, b_1\rangle) = a_2$, and so on. In this case, the play is uniquely defined by σ and $b = \langle b_0, b_1, b_2, \ldots\rangle$, so we write $\sigma * b = \langle a_0, b_0, a_1, b_1, \ldots\rangle$. We say that σ is a *winning strategy* for Alice in G_A if $\sigma * b \in A$ for all $b \in \mathbb{N}^{\mathbb{N}}$, that is, Alice wins by this strategy no matter how Bob plays. Similarly, we define a strategy for Bob as a function τ from finite sequences in $\mathbb{N}$ of odd length, to $\mathbb{N}$, etc. We say that the game G_A is *determined* if one of the players has a winning strategy.

Before we use such games to study the complexity of certain sets, it is worthwhile knowing of a few cases wherein we know games are determined. Recall that the topology on $\mathbb{N}^{\mathbb{N}}$ is generated by basic open sets of the form $U_s = \{x : s \sqsubseteq x\}$, where $s = \langle a_0, a_1, \ldots, a_k\rangle$, and $\sqsubseteq$ denotes extension.

Theorem 2. *(**Gale-Stewart**, 1953) Suppose $A \subseteq \mathbb{N}^{\mathbb{N}}$ is open (or closed), then G_A is determined.*

Proof: Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be open. Suppose that Alice does not have a winning strategy in G_A. We will show that, in this case, Bob has a winning strategy by always moving in such a way that he is "not losing yet". When Alice plays a_0, then because she does not have a winning strategy, there must be a b_0 such that Bob can still win from $\langle a_0, b_0\rangle$. Let Bob play b_0. When Alice plays a_1, again, because Bob can still win from $\langle a_0, b_0, a_1\rangle$, there must be a b_1 such that Bob can still win from $\langle a_0, b_0, a_1, b_1\rangle$, and so on. We claim that this is a winning strategy for Bob.

Let $x = \langle a_0, b_0, a_1, b_1, \ldots\rangle$ be a play in which Bob follows the above strategy. If $x \in A$, then since A is open, there is some $s = \langle a_0, b_0, a_1, b_1, \ldots, a_n, b_n\rangle$, an initial segment of x, such that $U_s \subseteq A$, in which case Bob has lost at the stage of the game which built s, a contradiction. Hence, $x \notin A$, and this is a winning strategy for Bob. The analogous statement for closed games is proven by interchanging the roles of Alice and Bob. Q.E.D.

Note that if we are only interested finite length games (with no allowed ties), the above proof also shows that they are determined, an older result due to Zermelo. Also, as a side note, the above proof does use some choice, namely that every nonempty pruned tree has a branch (i.e., the *Axiom of Dependent Choices*, which happens to be equivalent to the Baire Category Theorem).

While the Gale-Stewart Theorem has important consequences in its own right, such as its use in model theory for building isomorphisms via the back-and-forth method, we would like to see what we can learn about classes of sets for which determinacy of their associated games holds. Given the above theorem, it may be natural to formulate the following assertion:

Axiom of Determinacy (AD)**:** *For every subset $A \subseteq \mathbb{N}^{\mathbb{N}}$, the game G_A is determined.*

Recall that the *(Lebesgue) outer measure* $m^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ can be defined as

$$m^*(E) = \inf\{\sum_n \text{length}(I_n) : \text{each } I_n \text{ is an interval, } E \subseteq \bigcup_n I_n\}$$

where $E \subseteq \mathbb{R}$. A set $E \subseteq \mathbb{R}$ is *(Lebesgue) measurable* if for every $A \subseteq \mathbb{R}$, $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$, or alternatively, is contained in the σ-algebra $\mathcal{M}$ generated by the open sets and null (i.e., outer measure zero) sets. The restriction $m = m^* \restriction \mathcal{M}$ is called the *(Lebesgue) measure.*

Assuming AD, we can arrive at a very dramatic result:

Theorem 3. *(**Mycielski-Swierczkowski**, 1964)* AD *implies that every subset of $\mathbb{R}$ is Lebesgue measurable.*

In order to establish this result, we will need the following basic fact:

Lemma 4. *For any $E \subseteq \mathbb{R}$, there is a measurable set G such that $E \subseteq G$, $m^*(E) = m(G)$, and if Z is any measurable subset of $G \setminus E$, then Z is null.*

Proof: Left as an exercise for the 6110 students. Hint: Start with bounded E, in which case G can be taken to be a G_δ set. Q.E.D.

The meat of the proof of the theorem is contained in the following lemma:

Lemma 5. *Assuming* AD*, if $S \subseteq \mathbb{R}$ is such that every measurable subset $Z \subseteq S$ is null, then S is null.*

Proof: (of the Theorem from the Lemma) Let $E \subseteq \mathbb{R}$, and $G \supseteq E$ a measurable set such that if Z is any measurable subset of $G \setminus E$, then Z is null. Then, by the (yet unproven) lemma, $G \setminus E$ is null, hence measurable, and so $E = G \cup (G \setminus E)$ is also measurable. Q.E.D.

Proof: (of Lemma) It is clear that we may assume, without loss of generality, that $S \subseteq [0, 1]$. Fix $\epsilon > 0$. We will show that $m^*(S) \leq \epsilon$. Towards this end, we set up the *covering game.* If $\langle a_0, a_1, \ldots \rangle \in \mathbb{N}^{\mathbb{N}}$ is a sequence of 0s and 1s, let

$$a = \sum_{n=0}^{\infty} \frac{a_n}{2^{n+1}} \in [0, 1].$$

This allows us to (partially) translate from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{R}$. For each $n \in \mathbb{N}$, let G_k^n, $k = 0, 1, 2, \ldots$, be an enumeration of the (countable) collection $\mathcal{K}_n$ of all sets G such that (i) G is a union of finitely many open intervals with rational endpoints, and (ii) $m(G) \leq \epsilon / 2^{2n+1}$.

Informally, the rules of the covering game are that Alice is trying to play $a \in S$, while Bob is trying to cover a by $\bigcup_{n=0}^{\infty} H_n$ such that $H_n \in \mathcal{K}_n$ for all n. Formally, a play $\langle a_0, b_0, a_1, b_1, \ldots \rangle$ is won by Alice if (i) $a_n \in \{0,1\}$ for all n, (ii) $a \in S$, and (iii) $a \notin \bigcup_{n=0}^{\infty} G^n_{b_n}$. Bob wins otherwise. This encodes a (very complicated) target set in $\mathbb{N}^{\mathbb{N}}$, but since we are working under AD, that does not matter.

We claim that Alice does not have a winning strategy in this game. Suppose σ was such a strategy, then it is not hard to show that the function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ such that $f(\langle b_0, b_1, b_2, \ldots \rangle) = a$ where a, as above, is the result of Alice playing by σ, is a continuous map. It is a (non-trivial) fact (see Theorem 11.18 in [1]) that the continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$ in $\mathbb{R}$ is Lebesgue measurable, so in particular $Z = f(\mathbb{N}^{\mathbb{N}}) \subseteq \mathbb{R}$ is Lebesgue measurable. Moreover, $Z \subseteq S$, so by hypothesis, Z is null. But, a null set can be covered by a countable union $\bigcup_{n=0}^{\infty} H_n$ such that $H_n \in \mathcal{K}_n$ for all n, and therefore, if Bob plays $\langle b_0, b_1, b_2, \ldots \rangle$ where $G^n_{b_n} = H_n$, and Alice plays by σ, then Bob wins, contradicting out choice of σ. Hence Alice has no winning strategy.

By AD, this game is determined, so Bob must have a winning strategy, say τ. For each finite sequence $s = \langle a_0, a_1, \ldots, a_n \rangle$ of 0s and 1s, let $G_s \in \mathcal{K}_n$ be the set $G^n_{b_n}$ where $\langle b_0, b_1, \ldots, b_n \rangle$ are the moves that Bob plays by τ in response to $a_0, a_1, \ldots, a_n$. Since τ is a winning strategy, we must have that for every $a \in S$, $a \in \bigcup\{G_s : s \sqsubseteq a\}$, and hence,

$$S \subseteq \bigcup\{G_s : s \in \{0,1\}^{<\omega}\} = \bigcup_{n=1}^{\infty} \bigcup_{s \in \{0,1\}^n} G_s.$$

For every $n \geq 1$, if $s \in \{0,1\}^n$, then $m(G_s) \leq \epsilon/2^{2n}$, and so

$$m(\bigcup_{s \in \{0,1\}^n} G_s) \leq \epsilon/2^n.$$

Thus,

$$m^*(S) \leq m(\bigcup\{G_s : s \in \{0,1\}^{<\omega}\}) \leq \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon.$$

Hence $m^*(S) = 0$, as desired. Q.E.D.

But, it is a well-known fact, due to Vitali, that there are subset of $\mathbb{R}$ which are *not* Lebesgue measurable. Hence, AD is false within the usual axiomatic formulation of mathematics, namely ZFC. In fact, it is not overly hard to use the Axiom of Choice directly to produce a subset of $\mathbb{N}^{\mathbb{N}}$ whose corresponding game is not determined. However, this does not mean that determinacy is a hopeless cause.

In some sense we can think of determinacy as a kind of ultimate regularity property for a subset of $\mathbb{N}^{\mathbb{N}}$ (or $\mathbb{R}$) to satisfy. This is justified partially by the following collection of theorems, of which part (a) can be obtained by altering the proof given about only slightly:

Theorem 6. *Let* Γ *be an* adequate pointclass *in* $\mathbb{N}^{\mathbb{N}}$ *(that is, a class of subsets of* $\mathbb{N}^{\mathbb{N}}$ *which satisfy certain closure properties). If* G_A *is determined for* all $A \in \Gamma$, *then*

(a) ***(Mycielski-Swierczkowski**, 1964) every set in* Γ *is Lebesgue measurable (under a Borel correspondence between* $\mathbb{R}$ *and* $\mathbb{N}^{\mathbb{N}}$*);*

(b) ***(Banach-Oxtoby**, 1957) every set in* Γ *has the* property of Baire *(i.e., if* $A \in \Gamma$*, then there is an open set* $U \subseteq \mathbb{N}^{\mathbb{N}}$ *such that* $A \Delta U$ *is meager);*

(c) ***(Davis**, 1964) every set in* Γ *has the* perfect set property *(i.e., is either countable or contains a homeomorphic copy of the Cantor set).*

So, it is worthwhile considering how far we can push determinacy. The following *very* deep and difficult theorem (within ZFC), justifies determinacy for all *reasonably definable* sets:

Theorem 7. *(**Martin**, 1975) If $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Borel, then G_A is determined.*

In some sense, Martin's theorem is the upper limit. The next natural class of reasonably definable sets are the *projective sets*, namely those formed by taking continuous images of Borel sets (which may be restricted to projection maps $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, without loss of generality), and closing under complementation and continuous images (again, projections sufice). The simplest such, beyond the Borel sets themselves, are the *analytic sets*, continuous images of Borel sets, and their complements, the *co-analytic sets*.

In order to extend determinacy into this realm, we need additional axioms beyond ZFC which assert the existence of certain cardinal numbers, whose existence is not provable in ZFC, with certain desirable properties; these are the so-called *large cardinals axioms*. In particular, we have the following results:

Theorem 8. *(**Martin**, 1969) If there exists a measurable cardinal, then all analytic (and co-analytic) games are determined.*

For all projective games, we need more:

Theorem 9. *(**Martin-Steel**, 1989) If there exists infinitely many Woodin cardinals, then all projective games are determined.*

However, we have now entered into possibly shaky territory. Even at the level of analytic games, determinacy for such games is essentially equivalent to a large cardinal axiom. In particular, like all other large cardinal axioms, determinacy of analytic games (and hence projective games as well) implies the *relative (to* ZFC*) consistency* of "there is an *inaccessible cardinal*", the simplest kind of large cardinal, and this proof is carried out in ZFC. But, it is known that we can never prove this within ZFC, and hence not only can we not prove the determinacy of analytic games, we cannot even prove that this is consistent with the rest of mathematics!

References

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[2] Kechris, Alexander S. *Classical Descriptive Set Theory.* Graduate Texts in Mathematics. Springer, 1995, New York.