Turbulence and Classification

Iian Smythe

Department of Mathematics
Cornell University

Olivetti Club
September 30, 2014

Classification in Mathematics

What do we mean by "classification" in mathematics?
This may be best answered by examples.

Classification in Mathematics (cont'd)

From group theory:

Fundamental Theorem of Finitely Generated Abelian Groups

Every finitely generated abelian group G is isomorphic to

$$\mathbb{Z}^r \oplus (\mathbb{Z}/p_0^{n_0}\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/p_k^{n_k}\mathbb{Z})$$

for a unique positive integer r (the rank of G), and unique (up to order) prime powers $p_0^{n_0}, \ldots, p_k^{n_k}$ (the elementary divisors of G).

For G and H finitely generated abelian groups with rank and elementary divisors (listed in increasing order) $(r, p_0^{n_0}, \ldots, p_k^{n_k})$ and $(s, q_0^{m_0}, \ldots, q_l^{m_l})$ respectively,

$$G \cong H \quad \Longleftrightarrow \quad (r, p_0^{n_0}, \ldots, p_k^{n_k}) = (s, q_0^{m_0}, \ldots, q_l^{m_l}).$$

Classification in Mathematics (cont'd)

From geometric topology:

Classification of Orientable Surfaces

If M and N are closed connected orientable surfaces, then M and N are homeomorphic if and only if they have the same genus.

A more sophisticated example:

Mostow's Rigidity Theorem

If M and N are closed hyperbolic manifolds of dimension $n \geq 3$, then M and N are isometric if and only if they have isomorphic fundamental groups.

Classification in Mathematics (cont'd)

And from analysis:

Weyl–von Neumann Theorem

If S and T are bounded self-adjoint operators on ℓ^2, then S and T are unitarily equivalent modulo compact if and only if S and T have the same essential spectrum.

And ergodic theory:

Ornstein's Isomorphism Theorem

If $BS(\vec{p})$ and $BS(\vec{q})$ are Bernoulli shifts, then $BS(\vec{p})$ and $BS(\vec{q})$ are isomorphic if and only if they have the same entropy.

Classification in Mathematics (cont'd)

In each of the previous examples, we have the following:

- A class X of objects we wish to classify up to some notion of equivalence E.
- A set Y with another notion of equivalence F.
- And a map $f : X \to Y$ such that

$$x \, E \, y \quad \iff \quad f(x) \, F \, f(y)$$

  for all $x, y \in X$. Such a map is called a reduction of E to F.
- The $\Rightarrow$ direction above shows that objects in Y modulo F are invariants for objects in X modulo E, and when combined with the $\Leftarrow$ direction, they are complete invariants.

A general framework for classification

We will restrict our attention to only certain classes of objects, equivalence relations, and reductions. Why?

Example

Suppose that E is an equivalence relation on a set X, and Y a set such that $|X| \leq |Y|$ (in cardinality).
By the Axiom of Choice $|X/E| \leq |X|$, so there is an injection $X/E \to Y$.
This can be lifted to a map $f : X \to Y$ satisfying

$$x \, E \, y \quad \Longleftrightarrow \quad f(x) = f(y)$$

for all $x, y \in X$. i.e., f is a reduction of E to equality on Y.

This "construction" provides no insight into the classification problem, nor does it provide a practical way to compute these invariants.

A general framework for classification (cont'd)

To put it another way:
"The system of classification should assign invariants to points in the space X based on their intrinsic properties... we would have much greater respect for a system of classification that is fantastically difficult than one that just pulls down the axiom of choice and then goes home to bed." – Greg Hjorth (in *Classification and Orbit Equivalence Relations*, 2000)

Polish spaces and Borel sets

A Polish space is a completely metrizable separable topological space, e.g., $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, $[0,1]$, manifolds, separable Banach spaces, and countable products and closed (or G_δ) subsets of these.

Given such a space X, the collection $\mathcal{B}$ of all Borel subsets of X is the smallest σ-algebra generated by the open sets.

A subset A of a Polish space is analytic if it is a continuous image of a Borel set (in some other Polish space).

An equivalence relation E on a Polish space X is Borel (or analytic) if the set $E = \{(x,y) \in X^2 : x\ E\ y\}$ is Borel (or analytic) in X^2.

A function $f : X \to Y$ between Polish spaces is Borel if $f^{-1}(B)$ is Borel in X for every Borel set B in Y.

Borel reductions

The primary notion of our theory is as follows:

Given Borel (or analytic) equivalence relations E and F on Polish spaces X and Y respectively, a Borel reduction of E to F is a map $f : X \to Y$ which is Borel and satisfies

$$x \, E \, y \iff f(x) \, F \, f(y)$$

for all $x, y \in X$.

In this case, we say that E is Borel reducible to F, and write $E \leq_B F$.

If $E \leq_B F$ and $F \leq_B E$, then we say that E and F are Borel bireducible, and write $E \equiv_B F$.

Borel reductions (cont'd)

- Borel reducibility $\leq_B$ is a transitive ordering on equivalence relations; simply compose reductions.
- Thus, if $E \leq_B F$, then complete invariants for F yield complete invariants for E.
- $\leq_B$ is a measure of difficulty of classification for different equivalence relations.

Smooth equivalence relations

For a Polish space X, we denote by $\Delta(X)$ the equality relation on X.

We say that a Borel (or analytic) equivalence relation E is smooth if $E \leq_B \Delta(X)$ for some Polish X.

- Without loss of generality, we may assume in the above definition that $X = \mathbb{R}$, and thus smooth equivalence relations are exactly those which admit complete real number invariants.
- In some (rare) cases, E is nice enough so that X/E has the structure of a Polish space. Then, the quotient map $X \to X/E$ witnesses that E is smooth.

Program of (non-)classification

In our framework, the general program of completely classifying some class of objects up to a notion of equivalence is as follows:

- Define a Polish space X whose elements are canonical representatives of the objects.
- Show that the notion of equivalence can be realized as a Borel (or analytic) equivalence relation on X.
- Find a Borel (or analytic) equivalence relation F and a Borel reduction of E to F. Ideally, the equivalence relation F is better understood than E, e.g., $\Delta(\mathbb{R})$.

The general program of showing non-classifiability involves, in addition to the first two steps above:

- For a given Borel (or analytic) equivalence relation F, show that there can be no Borel reduction of E to F.

Spaces of structures

Example (Hyperspace of separable complete metric spaces)

Define a subset $\mathbb{X}$ of $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ consisting of elements $(r_{i,j})_{i,j\in\mathbb{N}}$ such that, for all $i,j,k \in \mathbb{N}$,

1. $r_{i,j} \geq 0$ and $r_{i,j} = 0$ if and only if $i = j$;
2. $r_{i,j} = r_{j,i}$;
3. $r_{i,j} \leq r_{i,k} + r_{k,j}$.

$\mathbb{X}$ is a G_δ subspace of $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ and thus Polish.
If (X,d) is an infinite separable complete metric space, say with dense subset $\{x_i : i \in \mathbb{N}\}$, then we identify it with the element $r_X \in \mathbb{X}$ given by

$$r_X = (r_{i,j})_{i,j\in\mathbb{N}} = (d(x_i,x_j))_{i,j\in\mathbb{N}}.$$

Isometry becomes an (analytic) equivalence relation $\cong_i$ on $\mathbb{X}$.

Similar constructions can be used to build spaces of manifolds, separable Banach spaces, etc.

Spaces of structures (cont'd)

Example (Space of countably infinite graphs)

Recall that a graph is a vertex set V together with an irreflexive and symmetric binary relation R on V. For countably infinite graphs, we may assume that $V = \mathbb{N}$, and thus such graphs can be identified with certain subsets $R \subseteq \mathbb{N}^2$.
Identify $\mathcal{P}(\mathbb{N}^2) = \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ via characteristic functions, and let X_γ be the collection of all $R = (R(n,m))_{n,m\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ such that for all $n, m \in \mathbb{N}$,

1. $R(m,m) = 0$;
2. $R(n,m) = R(m,n)$.

X_γ is a closed subspace of $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$, and thus Polish.
Isomorphism becomes an (analytic) equivalence relation $\cong_\gamma$ on X_γ.

Observe that $\cong_\gamma \leq_B \cong_i$, via the graph metric.

Classification by countable structures

Similar constructions can be used to construct spaces of countably infinite groups, rings, linear orders, etc, and the isomorphism relation of the corresponding structures becomes an analytic equivalence relation on that space.

This yields a very generous notion of "reasonably classifiable":

A Borel (or analytic) equivalence relation E is classifiable by countable structures if it is Borel reducible to the isomorphism relation for some class of countable first-order structures (e.g., graphs, groups, rings, linear orders, etc).

Fact

Smooth equivalence relations are classifiable by countable structures.

Polish group actions

Returning to the space of countably infinite graphs X_γ, let $R, S \in X_\gamma$, identified with their characteristic functions. As graphs, they are isomorphic if and only if there is a bijection $g : \mathbb{N} \to \mathbb{N}$ satisfying

$$R(n,m) = S(g(n), g(m)),$$

for all $n, m \in \mathbb{N}$.

This describes an action of the group $S_\infty \curvearrowright X_\gamma$ by

$$(g \cdot R)(n,m) = R(g^{-1}(n), g^{-1}(m))$$

for all $R \in X_\gamma$ and $n, m \in \mathbb{N}$. The orbit equivalence relation of this group action is exactly the isomorphism relation on X_γ.

Similarly, if X is a space of all countably infinite groups, rings, linear orders, etc, then the isomorphism relation is the orbit equivalence relation of an action of S_∞ on X.

Polish group actions (cont'd)

S_∞ is an example of Polish group, i.e., a topological group with a Polish topology, and the action $S_\infty \curvearrowright X_\gamma$ is a continuous action of this group on the Polish space X_γ.

In general, if G is a Polish group acting continuously on a Polish space X, the orbit equivalence relation E_G is given by

$$x \, E_G \, y \quad \Longleftrightarrow \quad \exists g \in G(g \cdot x = y),$$

and is an analytic equivalence relation on X.

- Note that the trivial action of $G = \{e\}$ induces $\Delta(X)$.
- In fact, the class of analytic equivalence relations induced by Polish group actions is very large.

Review of Baire category notions

Recall that a subset of M of a Polish space X is meager (or first category) if it is contained in a countable union of (closed) nowhere dense sets. A set $C \subseteq X$ is comeager if $X \setminus C$ is meager.

Baire Category Theorem

If X is a Polish space, then X is not meager in itself. In fact, if M is meager in X, then $X \setminus M$ is dense in X.

Thus, meager is a good topological notion of "small", and comeager a notion of "large", somewhat analogous to null and almost-everywhere in measure theory.

Review of Baire category notions (cont'd)

On a Polish space X, we define a new quantifier $\forall^*$ as follows:

If A is a property of elements of X,

$$\forall^* x A(x) \quad \text{means} \quad \{x \in X : A(x)\} \text{ is comeager in } X,$$

i.e., A occurs for "comeagerly many" $x \in X$.

Theorem (Kuratowski–Ulam)

Let X and Y be Polish spaces and $A \subseteq X \times Y$ an analytic subset of $X \times Y$. Then,

$$\forall^*(x, y) A(x, y) \iff \forall^* x \forall^* y A(x, y) \iff \forall^* y \forall^* x A(x, y).$$

Generic ergodicity

Let G be a Polish group acting continuously on a Polish space X. This action is said to be generically ergodic if every G-invariant Borel subset of X is either meager or comeager.

Proposition

Let G be a Polish group acting continuously on a Polish space X. The following are equivalent:

1. *The action is generically ergodic.*
2. *There is a G-invariant dense G_δ set $Y \subseteq X$ all of whose G-orbits are dense in X.*
3. *There is a dense G-orbit.*

In particular, if every orbit of a continuous action is dense, then it is generically ergodic.

Generic ergodicity (cont'd)

Proposition

Let G be a Polish group acting generically ergodically on a Polish space X, and $f : X \to \mathbb{R}$ a G-invariant Borel map, that is,

$$x \, E_G \, y \quad \Rightarrow \quad f(x) = f(y)$$

for all $x, y \in X$. Then, f is constant on a comeager set $C \subseteq X$.

Proof.

By generic ergodicity, there is a G-invariant dense G_δ set $Y \subseteq X$ all of whose orbits under G are dense.
Since the map $f : X \to \mathbb{R}$ is Borel, it is continuous on a comeager set $C \subseteq Y$. We claim that f is constant on C.

Generic ergodicity (cont'd)

Proof.

For each $g \in G$, the map $y \mapsto g \cdot y$ is a homeomorphism of Y, and thus, there is a comeager set in Y which gets mapped into C by g, i.e.,

$$\forall g \in G \forall^* x \in Y (g \cdot x \in C).$$

By Kuratowski–Ulam,

$$\forall^* x \in Y \forall^* g \in G (g \cdot x \in C).$$

So, there is an $x_0 \in Y$ such that $\forall^* g \in G(g \cdot x_0 \in C)$. We claim that $[x_0]_G \cap C$ is dense in Y. Let $U \subseteq Y$ be nonempty open. Since $[x_0]_G$ is dense, the set $\{g \in G : g \cdot x_0 \in U\}$ is nonempty and open, and thus intersects the comeager set $\{g \in G : g \cdot x_0 \in C\}$. i.e., $[x_0]_G \cap C \cap U \neq \emptyset$. We claim that f takes the constant value $f(x_0)$ on C. Take $z \in C$. By density of $[x_0]_G \cap C$ in Y, there is $(z_n)_n$ in $[x_0]_G \cap C$ such that $z_n \to z$. But continuity and G-invariance of f imply $f(z) = \lim_n f(z_n) = f(x_0)$. □

Generic ergodicity (cont'd)

Corollary

Let G be a Polish group acting generically ergodically on a Polish space X such that every orbit is meager. Then, E_G is not smooth.

Proof.

If E_G was smooth, then the Borel reduction $f : X \to \mathbb{R}$ is G-invariant. By the previous proposition, f is constant on a comeager set, but since f is a reduction, this comeager set is contained in a single orbit. □

Corollary

Let $\mathbb{Q}$ act on $\mathbb{R}$ by translation. Then, the orbit equivalence relation $E_{\mathbb{Q}}$ is not smooth. Consequently, if E is an analytic equivalence relation and $E_{\mathbb{Q}} \leq_B E$, then E is not smooth.

Generic ergodicity (cont'd)

- One way to rephrase this result: If $G \curvearrowright X$ is generically ergodic, and we let the trivial group $\{e\}$ act on $\mathbb{R}$, then any Borel equivariant map $X \to \mathbb{R}$ maps a comeager set to a single $\{e\}$-orbit.
- Recall that an equivalence relation is classifiable by countable structures if it is Borel reducible to the isomorphism relation for some class of countable structures, and the latter equivalence relations are always induced by actions of S_∞.
- Thus, if we can isolate a condition of Polish group actions $G \curvearrowright X$ such that whenever S_∞ acts on a Polish space Y, any Borel equivariant map $X \to Y$ maps a comeager set to a single S_∞-orbit, we will obtain a criterion for when certain equivalence relations are not classifiable by countable structures.

Turbulence

Such a condition was isolated by Greg Hjorth in the late 1990's.

We need a technical definition:
Let G be a Polish group acting on a Polish space X. For $U \subseteq X$ open, and $V \subseteq G$ a symmetric open neighborhood of the identity e_G, the (U, V)-local orbit of a point $x \in U$, denoted by $\mathcal{O}(x, U, V)$, is the set of all $y \in U$ such that there are $x = x_0, x_1, \ldots, x_n = y$ in U, and $g_0, \ldots, g_{n-1} \in V$ for which

$$x_{i+1} = g_i \cdot x_i \quad \text{whenever } i < n.$$

($\longleftarrow$) See picture on side board.

Turbulence (cont'd)

For such $G \curvearrowright X$ as usual, we say that the action of G is turbulent if

- every orbit is dense;
- every orbit is meager;
- every (U, V)-local orbit is somewhere dense, i.e., for every U and V as above, and every $x \in U$, $\overline{\mathcal{O}(x, U, V)}$ has nonempty interior.

Theorem (Hjorth)

Let G be a Polish group acting turbulently on a Polish space X, and suppose that S_∞ acts continuously on a Polish space Y. If $f : X \to Y$ is an equivariant Borel map, that is,

$$x \, E_G \, y \quad \Rightarrow \quad f(x) \, E_{S_\infty} \, f(y),$$

for all $x, y \in X$, then f maps a comeager set $C \subseteq X$ to a single S_∞-orbit.

Turbulence (cont'd)

Corollary

Let G be a Polish group acting turbulently on a Polish space X. Then, E_G is not classifiable by countable structures.

Proof.

If E_G was classifiable by countable structures, then there is a Borel reduction f of E_G to E_{S_∞}, for some continuous action of S_∞.
f is equivariant, so by Hjorth's Theorem, there is a comeager set $C \subseteq X$ such that f maps C to a single S_∞-orbit.
But since f is a reduction, C must be contained in a single G-orbit. □

Corollary

Let G be a Polish group acting turbulently on a Polish space X, and E an analytic equivalence relation. If $E_G \leq_B E$, then E is not classifiable by countable structures.

Examples of turbulent actions

Example

Clearly S_∞ can never act turbulently. Neither can locally compact Polish groups (e.g., discrete groups, compact groups, Lie groups, etc).

Example

Let G be a proper Borel subgroup of $(\mathbb{R}^{\mathbb{N}}, +)$ which is Polish in some topology, and such that for every $\vec{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n$, there is a $g \in G$ which agrees with $\vec{x}$ on its first n coordinates, e.g., c_0 and ℓ^p $(1 \leq p < \infty)$. Then the action of G by translation on $\mathbb{R}^{\mathbb{N}}$ is turbulent.

Example

If X is a separable infinite dimensional Banach space, and Y a Borel linear subspace of X which is dense and Polish in some topology, then the action of Y on X by translation is turbulent.

Examples of non-classifiability

Using Hjorth's theory of turbulence, a number of non-classification results have been produced in the last 15 years:

Theorem

The following equivalence relations are not classifiable by countable structures.

- *(Hjorth) Isomorphism of measure preserving transformations.*
- *(Hjorth) Conjugation of homeomorphisms of* $[0, 1]^2$.
- *(Hjorth–Kechris) Biholomorphism of complex manifolds of dimension* ≥ 2.
- *(Kechris–Sofronidis) Unitary equivalence of self-adjoint operators.*
- *(Farah–Toms–Törnquist) Isomorphism of simple separable unital C*-algebras.*

$\vdots$

Thanks for listening!