MATH 1110 (Lecture 002)
August 30, 2013

Pre-Calculus Review Problems — Solutions

1 Algebra and Geometry

Problem 1. Give equations for the following lines in *both* point-slope and slope-intercept form.
(a) The line which passes through the point $(1, 2)$ having slope 4.
(b) The line which passes through the points $(-1, 1)$ and $(2, -1)$.
(c) The line parallel to $y = \frac{1}{2}x + 2$, with y-intercept $(0, -1)$.
(d) The line perpendicular to $y = -3x + 1$ which passes through the origin.

Solution: (a) The point-slope form is

$$y - 2 = 4(x - 1).$$

Solving for y,

$$\begin{aligned} y &= 4(x-1) + 2 \\ &= 4x - 4 + 2 \\ &= 4x - 2, \end{aligned}$$

yields the slope-intercept form,

$$y = 4x - 2.$$

(b) First, we compute the slope using the familiar "rise-over-run" formula,

$$m = \frac{-1 - 1}{2 - (-1)} = -\frac{2}{3}.$$

The point-slope form (using the first point) is,

$$y - 1 = -\frac{2}{3}(x + 1),$$

and solving for y yields the slope-intercept form,

$$y = -\frac{2}{3}x + \frac{1}{3}.$$

(c) The slope of our desired line is $\frac{1}{2}$, since parallel lines must have the same slope. The point-slope form is,

$$y - (-1) = \frac{1}{2}(x - 0),$$

and the slope-intercept form is

$$y = \frac{1}{2}x - 1.$$

(d) The slope of our desired line is $\frac{1}{3}$, since it must be the negative reciprocal of the slope any line to which it is perpendicular. The point-slope form is,

$$y - 0 = \frac{1}{3}(x - 0),$$

and the slope-intercept form is,

$$y = \frac{1}{3}x.$$

Problem 2. Find the point of intersection, if there is one, between the following lines:
(a) $y = -x + 5$ and $y - 2 = 3(x + 1)$
(b) The line passing through $(-1, -2)$ and the origin, and the line $y = 2x - 2$.

Solution: (a) First, we write both lines in slope-intercept form,

$$y = -x + 5 \qquad y = 3x + 5.$$

If (x, y) is a point of intersection of the lines, it must satisfy both equations. Assuming (x, y) is as such, we have that

$$\begin{aligned} -x + 5 &= 3x + 5 \\ -x &= 3x \\ x &= 0. \end{aligned}$$

Thus, $x = 0$. To find y, we can plug $x = 0$ into either one of the original equations, and get that $y = 5$. Thus, $(0, 5)$ is the (unique) point of intersection.

(b) The line passing through $(-1, -2)$ and the origin has slope

$$m = \frac{0 - (-2)}{0 - (-1)} = 2,$$

and can be expressed by the equation $y = 2x$. But, this line is parallel to (and distinct from) the line $y = 2x - 2$, so they cannot have any points of intersection.

Problem 3. Find all real roots x of the following polynomials, and factor into irreducible polynomials.
(a) $6x^2 + 5x + 1$
(b) $-x^2 + x + 1$
(c) $2x^2 - 3x + 5$
(d) $x^3 + 6x^2 - 7x$
(e) $x^3 - x^2 + x - 1$
(f) $x^4 - 2x^2 + 1$

Solution: Note that a polynomial is *irreducible* if it cannot be factored into non-constant polynomials with real coefficients.
(a)

$$\begin{aligned} 6x^2 + 5x + 1 &= 6x^2 + 3x + 2x + 1 \\ &= 3x(2x + 1) + 1(2x + 1) \\ &= (3x + 1)(2x + 1). \end{aligned}$$

This factors the polynomial into irreducibles, and shows that its roots are $x = -\frac{1}{3}$ and $x = -\frac{1}{2}$.

(b) We use the quadratic formula:

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1^2 - 4(-1)(1)}}{2(-1)} \\ &= \frac{-1 \pm \sqrt{1 + 4}}{-2} \\ &= \frac{1 \pm \sqrt{5}}{2}, \end{aligned}$$

Thus, $x = \frac{1 \pm \sqrt{5}}{2}$ are the two real roots of the polynomial. It follows that the polynomial factors as

$$-x^2 + x + 1 = -\left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right)$$

(c) We use the quadratic formula:

$$\begin{aligned} x &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(5)}}{2(2)} \\ &= \frac{3 \pm \sqrt{-31}}{4}, \end{aligned}$$

which cannot be real. Thus, the polynomial has no real roots, and cannot be factored further (a polynomial of degree 2 or 3 is irreducible if and only if it has no roots).

(d)

$$\begin{aligned} x^3 + 6x^2 - 7x &= x(x^2 + 6x - 7) \\ &= x(x^2 - x + 7x - 7) \\ &= x(x(x-1) + 7(x-1)) \\ &= x(x+7)(x-1) \end{aligned}$$

This factors the polynomial into irreducibles, and shows that the its roots are $x = 0$, $x = -7$ and $x = 1$.

(e) It is easy to see that $x^3 - x^2 + x - 1$ has root $x = 1$, since

$$(1)^3 - (1)^2 + 1 - 1 = 1 - 1 = 0.$$

So, we can factor out an $(x - 1)$. Using polynomial long division,

$$\begin{array}{r} x^2 \qquad +1 \\ x-1 \overline{\big)\ x^3 - x^2 + x - 1} \\ \underline{-x^3 + x^2 \qquad\quad} \\ x - 1 \\ \underline{-x + 1} \\ 0 \end{array}$$

we get that

$$x^3 - x^2 + x - 1 = (x-1)(x^2+1),$$

and $x^2 + 1$ has no real roots since $x^2 + 1 > 0$ for all $x \in \mathbb{R}$. Thus, this factors the polynomial into irreducibles, and the only real root is $x = 1$.

(f) Let $z = x^2$, then

$$\begin{aligned} x^4 - 2x^2 + 1 &= z^2 - 2z + 1 \\ &= (z-1)(z-1) \\ &= (x^2-1)(x^2-1) \\ &= (x-1)(x+1)(x-1)(x+1). \end{aligned}$$

This factors the polynomial into irreducibles, and shows that its roots are $x = \pm 1$.

Problem 4. Solve the following equations for x.

(a) $3\sqrt{x} = x - 4$

(b) $\sqrt{x+2} + \sqrt{x-2} = \sqrt{4x-2}$

(c) $x = 4\sqrt[3]{x}$.

(d) $\frac{x-1}{x-2} + \frac{2x+1}{x+2} = 0$

Solution: (a) First, note that the presence of $\sqrt{x}$ means that any solutions x must be ≥ 0.

$$\begin{aligned} 3\sqrt{x} &= x - 4 \\ 9x &= (x-4)^2 \\ 9x &= x^2 - 8x + 16 \\ 0 &= x^2 - 17x + 16 \\ 0 &= (x-16)(x-1). \end{aligned}$$

The solutions the last equation are $x = 1$ and $x = 16$, and since these are both positive, they are our solutions.

(b) The presence of $\sqrt{x+2}$, $\sqrt{x-2}$ and $\sqrt{4x-2}$ means that any solution x must satisfy $x \geq -2$, $x \geq 2$, and $x \geq \frac{1}{2}$, but the first and third of these are redundant, so it suffices to look for solutions with $x \geq 2$.

$$\begin{aligned} \sqrt{x+2} + \sqrt{x-2} &= \sqrt{4x-2} \\ (\sqrt{x+2} + \sqrt{x-2})^2 &= 4x - 2 \\ (x+2) + 2\sqrt{x+2}\sqrt{x-2} + (x-2) &= 4x - 2 \\ 2\sqrt{x+2}\sqrt{x-2} + 2x &= 4x - 2 \\ 2\sqrt{x+2}\sqrt{x-2} &= 2x - 2 \\ 4(x+2)(x-2) &= (2x-2)^2 \\ 4(x^2-4) &= 4x^2 - 8x + 4 \\ 4x^2 - 16 &= 4x^2 - 8x + 4 \\ -20 &= -8x \\ \frac{5}{2} &= x. \end{aligned}$$

Note that $x = \frac{5}{2} \geq 2$, as required, so this is the solution.

(c) Every real number has a cube root, so $\sqrt[3]{x}$ does not impose any restrictions on our solution. Clearly $x = 0$ is a solution, so in the following derivation, we can assume that $x \neq 0$.

$$\begin{aligned} x &= 4\sqrt[3]{x} \\ x^3 &= 64x \\ x^2 &= 64 \quad (\text{since } x \neq 0) \\ x &= \pm 8. \end{aligned}$$

Thus, $x = 0$ and $x = \pm 8$ are the solutions.

(d) Note that any solution x *cannot* be equal to 2 or -2.

$$\begin{aligned}
\frac{x-1}{x-2} + \frac{2x+1}{x+2} &= 0 \\
\frac{x-1}{x-2}\cdot\frac{x+2}{x+2} + \frac{2x+1}{x+2}\cdot\frac{x-2}{x-2} &= 0 \\
\frac{x^2+x-2}{x^2-4} + \frac{2x^2-3x-2}{x^2-4} &= 0 \\
\frac{3x^2-2x-4}{x^2-4} &= 0.
\end{aligned}$$

The only way for this equation to be true is if the numerator on the left-hand side is 0, which occurs exactly when x is a root of $3x^2 - 2x - 4$. We use the quadratic formula,

$$x = \frac{2 \pm \sqrt{4-4(3)(-4)}}{6} = \frac{1 \pm \sqrt{1+12}}{3} = \frac{1 \pm \sqrt{13}}{3}.$$

Since neither of these solutions are equal to 2 or -2, we have that these are the solutions to original equation.

Problem 5. Find the equations of the following shapes.
(a) A circle of radius 2, centered at $(1, 2)$.
(b) A circle centered at the origin, and tangent to the line $y = -2x + 2$.

Solution: (a) The equation of the circle is

$$(x-1)^2 + (y-2)^2 = 4.$$

(b) [**This is trickier. If you couldn't do this problem, that is okay!**] Since the circle is centered at the origin and tangent to $y = -2x + 2$, it must intersect $y = -2x + 2$ at the point on this line which is nearest to the origin. This is given by the intersection of $y = -2x + 2$ with the perpendicular line $y = \frac{1}{2}x$ through the origin. We can find their intersection,

$$\begin{aligned}
-2x + 2 &= \frac{1}{2}x \\
2 &= \frac{5}{2}x \\
\frac{4}{5} &= x.
\end{aligned}$$

Plugging this in for x in the equation $y = \frac{1}{2}x$ yields $y = \frac{2}{5}$, so the point of intersection is $(\frac{4}{5}, \frac{2}{5})$. The radius r of our circle is the distance from the origin to the point $(\frac{4}{5}, \frac{2}{5})$, and so

$$r^2 = \left(\frac{4^2}{5^2} + \frac{2^2}{5^2}\right) = \frac{20}{25} = \frac{4}{5}.$$

Thus, the equation of the circle is

$$x^2 + y^2 = \frac{4}{5}.$$

2 Exponents and Logarithms

Problem 6. Simplify the following expressions.
(a) $\frac{x^2(x^3)^4}{x^4}$
(b) $9^{1/3} \cdot 9^{1/6}$
(c) $(\sqrt{3})^{1/2} \cdot (\sqrt{12})^{1/2}$

Solution: (a)

$$\frac{x^2(x^3)^4}{x^4} = \frac{x^2(x^{12})}{x^4} = \frac{x^{14}}{x^4} = x^{10}.$$

(b)

$$9^{1/3} \cdot 9^{1/6} = 9^{1/3+1/6} = 9^{3/6} = 9^{1/2} = 3.$$

(c)

$$(\sqrt{3})^{1/2} \cdot (\sqrt{12})^{1/2} = (\sqrt{3})^{1/2} \cdot (\sqrt{4 \cdot 3})^{1/2} = (\sqrt{3})^{1/2} \cdot (2\sqrt{3})^{1/2} = \sqrt{2}(\sqrt{3})^{1/2}(\sqrt{3})^{1/2} = \sqrt{2}\sqrt{3} = \sqrt{6}.$$

Problem 7. Simplify the following expressions.
(a) $\log_9(3)\log_5(1/25)$
(b) $\ln(\ln(e)) + \log_2(8)$
(c) $2\ln(3x-4) - 5\ln(2x-7)$ (write as an expression containing a single logarithm)

Solution: (a)

$$\log_9(3)\log_5(1/25) = (1/2)(-2) = -1.$$

(b)

$$\ln(\ln(e)) + \log_2(8) = \ln(1) + 3 = 0 + 3 = 3.$$

(c)

$$2\ln(3x-4) - 5\ln(2x-7) = \ln((3x-4)^2) - \ln((2x-7)^5) = \ln\left(\frac{(3x-4)^2}{(2x-7)^5}\right).$$

3 Inequalities

Problem 8. Solve for x in the following inequalities, i.e., find the set of all x which satisfy the given inequality.
(a) $5x - 3 \leq 7 - 3x$
(b) $|3x - 7| < 4$
(c) $(x-1)^2 < 9$
(d) $\sqrt{x-1} \geq 2$.

Solution: (a)

$$\begin{aligned} 5x - 3 &\leq 7 - 3x \\ 5x + 3x &\leq 7 + 3 \\ 8x &\leq 10 \\ x &\leq 5/4. \end{aligned}$$

Thus, the solution is $\{x : x \leq 5/4\}$, or equivalently, $(-\infty, 5/4]$.

(b) Due to the absolute value, there are two cases to consider: If $3x - 7 \geq 0$, then we have

$$0 \leq 3x - 7 < 4.$$

So,

$$\begin{aligned} 0 \leq 3x - 7 &< 4 \\ 7 \leq 3x &< 11 \\ 7/3 \leq x &< 11/3. \end{aligned}$$

Thus, the solution in this case is $\{x : 7/3 \leq x < 11/3\}$, or $[7/3, 11/3)$.

If $3x - 7 < 0$, then we have

$$0 < -(3x - 7) < 4.$$

So,

$$\begin{aligned} 0 < -(3x - 7) &< 4 \\ 0 > 3x - 7 &> -4 \\ 7 > 3x &> 3 \\ 7/3 > x &> 1. \end{aligned}$$

Thus, the solution in this case is $\{x : 1 < x < 7/3\}$, or $(1, 7/3)$.

The solution in general is the union of these two sets, namely $\{x : 7/3 \leq x < 11/3\} \cup \{x : 1 < x < 7/3\}$, which is just $\{x : 1 < x < 11/3\}$, or $(1, 11/3)$.

(c) Note that $(x - 1)^2 = |x - 1|^2$, so $(x - 1)^2 < 9$ implies that

$$|x - 1| < 3.$$

There are two cases to consider: If $x - 1 \geq 0$, then

$$\begin{aligned} 0 \leq x - 1 &< 3 \\ 1 \leq x &< 4. \end{aligned}$$

Thus, the solution in this case is $\{x : 1 \leq x < 4\}$. If $x - 1 < 0$, then

$$\begin{aligned} 0 < -(x - 1) &< 3 \\ 0 > x - 1 &> -3 \\ 1 > x &> -2. \end{aligned}$$

Thus the solution in this case is $\{x : -2 < x < 1\}$. Thus, the solution in general is $\{x : -2 < x < 4\}$, or $(-2, 4)$.

(d) The presence of $\sqrt{x - 1}$ tells us that the solution must be contained in $[1, \infty)$, i.e., any x in the solution set is ≥ 1.

$$\begin{aligned} \sqrt{x - 1} &\geq 2 \\ x - 1 &\geq 4 \\ x &\geq 5. \end{aligned}$$

Thus, the solution is $\{x : x \geq 5\}$, or $[5, \infty)$.

4 Trigonometry

Problem 9. Fill in the following table with *exact* values:

θ in degrees	θ in radians $0 \le \theta < 2\pi$	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	0	1	0
$30°$	$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3} = \sqrt{3}/3$
$45°$	$\pi/4$	$1/\sqrt{2} = \sqrt{2}/2$	$1/\sqrt{2} = \sqrt{2}/2$	1
$60°$	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$90°$	$\pi/2$	1	0	not defined
$120°$	$2\pi/3$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$
$135°$	$3\pi/4$	$1/\sqrt{2} = \sqrt{2}/2$	$-1/\sqrt{2} = -\sqrt{2}/2$	-1
$150°$	$5\pi/6$	$1/2$	$-\sqrt{3}/2$	$-1\sqrt{3} = -\sqrt{3}/3$
$180°$	π	0	-1	0
$210°$	$7\pi/6$	$-1/2$	$-\sqrt{3}/2$	$1/\sqrt{3} = \sqrt{3}/3$
$225°$	$5\pi/4$	$-1/\sqrt{2} = -\sqrt{2}/2$	$-1/\sqrt{2} = -\sqrt{2}/2$	1
$240°$	$4\pi/3$	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$
$270°$	$3\pi/2$	-1	0	not defined
$300°$	$5\pi/3$	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$
$315°$	$7\pi/4$	$-1/\sqrt{2} = -\sqrt{2}/2$	$1/\sqrt{2} = \sqrt{2}/2$	-1
$330°$	$11\pi/6$	$-1/2$	$\sqrt{3}/2$	$-1/\sqrt{3} = -\sqrt{3}/3$

Problem 10. Find the exact values for the following expressions.
(a) $\tan\theta$ when θ is in the third quadrant and $\sin\theta = \frac{4}{5}$. [**Note: There was a typo here! It should be** $\sin\theta = -\frac{4}{5}$**.**]
(b) $\sin\frac{\pi}{12}$. (Hint: remember your trig identities?)

Solution: (a) Recall that when θ is in standard position relative to the xy-axis,

$$\sin\theta = \frac{x_0}{r},$$

where the point (x_0, y_0) is the intersection of the terminal ray of the angle with a circle centered at the origin and having radius r. Since θ lies in the third quadrant, the x_0 must be negative, and we can take it to be -4, and with $r = 5$. Since,

$$\cos\theta = \frac{y_0}{r},$$

we must find y_0. By Pythagoras, the length y_0 is 3 (this is a 3-4-5-right triangle), and it is negative, since θ is in the third quadrant. That is, $y_0 = -3$. Thus,

$$\tan\theta = \frac{x_0}{y_0} = \frac{-4}{-3} = \frac{4}{3}.$$

(b) First, note that $\pi/12 = \pi/3 - \pi/4$. We can use the following trig identity:

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi.$$

Taking $\theta = \pi/3$ and $\phi = \pi/4$, we get that

$$\begin{aligned}
\sin(\pi/12) &= \sin(\pi/3)\cos(\pi/4) - \cos(\pi/3)\sin(\pi/4) \\
&= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{6} - \sqrt{2}}{4}.
\end{aligned}$$