Set Theory and Projections in the Calkin Algebra

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Abstract

We discuss the partial order of projections in the Calkin algebra, or equivalently, the projection operators on a Hilbert space modulo compact perturbation. Analogies are drawn to the collection of subsets of the natural numbers modulo finite. Particular focus is placed on the gap structure of both orders, and the role of set theory within this investigation. These notes were made to accompany the author's Admission to Candidacy (A) Exam at Cornell University.

1 The setting

1.1 $\mathcal{P}(\omega)$ modulo finite

Throughout, $\omega = \{0, 1, 2, \ldots\}$ will denote the set of natural numbers, and $\mathcal{P}(\omega)$ its power set. Via characteristic functions, we identify $\mathcal{P}(\omega)$ with the *Cantor space* 2^ω (in the usual product topology, with $2 = \{0, 1\}$ discrete), thus endowing $\mathcal{P}(\omega)$ with the structure of a compact Polish space, with *Borel* subsets being those in the σ-algebra generated by the open sets, and *analytic* subsets being continuous images of Borel subsets (from possibly other Polish spaces).

Notice that $\mathcal{P}(\omega)$ is partially ordered by $\subset$, and together with $\cap$ and $\cup$, this gives $\mathcal{P}(\omega)$ the structure of a complete boolean algebra (taking $+$ to be symmetric difference $\triangle$, $\times$ to be $\cap$, $0 = \emptyset$, and $1 = \omega$, $\mathcal{P}(\omega)$ forms a boolean ring).

Definition 1.1. $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an *ideal* if
(i) $\emptyset \in \mathcal{I}$,
(ii) for every $x, y \in \mathcal{I}$, $x \cup y \in \mathcal{I}$, and
(iii) for every $x \in \mathcal{I}$ and $y \subseteq \omega$, if $y \subseteq x$, then $y \in \mathcal{I}$.
(Note that this is equivalent to $\mathcal{I}$ being an ideal in the ring $(\mathcal{P}(\omega), \triangle, \cap, \emptyset, \omega)$.) We say that an ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is *Borel*, *analytic*, etc, if it has this property as a subset of $\mathcal{P}(\omega)$.

Intuitively, an ideal gives a notion of *smallness* for subsets of ω. The most important example of a non-trivial proper ideal in $\mathcal{P}(\omega)$ is the ideal of *finite sets* (or *Fréchet ideal*):

$$\mathrm{Fin} = \{x \subset \omega : x \text{ is finite}\}.$$

Note that Fin is countable, and in particular, Borel.

Whenever we are given an ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$, we can consider the quotient partial order (or boolean algebra) $\mathcal{P}(\omega)/\mathcal{I}$ with the corresponding order defined on equivalence classes modulo $\mathcal{I}$. In order to avoid dealing directly with equivalence classes, we instead pull back this structure to $\mathcal{P}(\omega)$ by modifying $\subset$ to $\subset_{\mathcal{I}}$:

$$x \subset_{\mathcal{I}} y \quad \text{if and only if} \quad x \setminus y \in \mathcal{I} \text{ and } y \setminus x \notin \mathcal{I}.$$

We write $x \subseteq_{\mathcal{I}} y$ if $x \setminus y \in \mathcal{I}$, and $x \equiv_{\mathcal{I}} y$ if $x \subseteq_{\mathcal{I}} y$ and $y \subseteq_{\mathcal{I}} y$ (i.e., x and y are equivalent modulo $\mathcal{I}$). Then, $\subset_{\mathcal{I}}$ is a (not necessarily antisymmetric) partial order on $\mathcal{P}(\omega)$, and studying $(\mathcal{P}(\omega), \subset_{\mathcal{I}})$ is more-or-less equivalent to studying $\mathcal{P}(\omega)/\mathcal{I}$.

In the particular case of $\mathcal{I} = \text{Fin}$, we instead write $x \subset^* y$, $x \subseteq^* y$ (x is *almost contained* in y), $x \equiv^* y$ (x is *almost equal* to y), etc, so studying $(\mathcal{P}(\omega), \subset^*)$ is equivalent to studying $\mathcal{P}(\omega)/\text{Fin}$. If $x \cap y \in \text{Fin}$, we say that x and y are *almost disjoint*.

There is a particular class of ideals in $\mathcal{P}(\omega)$ that will be relevant for our study:

Definition 1.2. An ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is a *P-ideal* if it is σ-directed by $\subseteq^*$, i.e., for every countable collection C of elements of $\mathcal{I}$, there is a $y \in \mathcal{I}$ such that for all $c \in C$, $c \subseteq^* y$.

Note that Fin is (trivially) a P-ideal.

1.2 $\mathcal{P}(H)$ modulo compact perturbation

Throughout, let H be an infinite dimensional separable complex Hilbert space, e.g., $H = \ell^2(\omega)$, with inner product $\langle \cdot, \cdot \rangle$ (recall that all such spaces are unitarily isomorphic). $\mathcal{B}(H)$ will denote the Banach space of bounded linear operators on H, with the operator norm. The adjoint operation * on $\mathcal{B}(H)$ is defined by, for $T \in \mathcal{B}(H)$,

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \quad \text{for all } x, y \in H.$$

Together with composition of operators, this endows $\mathcal{B}(H)$ with the structure of a C*-algebra. Standard references for the relevant operator/C*-algebra theory are [24], [23] and [3].

Recall that if X is a *closed* subspace of H, then

$$X^\perp = \{y \in H : \forall x \in X(\langle x, y \rangle = 0)\}$$

is also a closed subspace of H, and every $y \in H$ can be written uniquely as $y = x + x^\perp$, where $x \in X$ and $x^\perp \in X^\perp$. The map $P : H \to H$ given by $y \mapsto x$ is called the *(orthogonal) projection* onto X. P is easily seen to be a bounded linear operator ($\|P\| = 1$ whenever $P \neq 0$) with $\text{ran}(P) = X$, and a *self-adjoint idempotent*, i.e., $P^2 = P^* = P$. In fact, every self-adjoint idempotent $P \in \mathcal{B}(H)$ is the projection onto a closed subspace of H, namely $\text{ran}(P)$.

Definition 1.3. An operator $P \in \mathcal{B}(H)$ is a *projection* if it is a self-adjoint idempotent. We let $\mathcal{P}(H)$ denote the collection of all projections on H. More generally, an element p of a C*-algebra $\mathcal{A}$ is a *projection* if it is a self-adjoint idempotent.

Note that the natural correspondence between closed subspaces of H and projections endows $\mathcal{P}(H)$ with a partial ordering:

$$P \leq Q \quad \text{if and only if} \quad \text{ran}(P) \subseteq \text{ran}(Q)$$

and the structure of a (complete) orthocomplemented lattice:

$$P \wedge Q = \text{projection onto } \operatorname{ran}(P) \cap \operatorname{ran}(Q)$$

$$P \vee Q = \text{projection onto } \overline{\operatorname{ran}(P) + \operatorname{ran}(Q)}$$

$$P^{\perp} = \text{projection onto } \operatorname{ran}(P)^{\perp} = I - P$$

The following facts are easy to verify:

Proposition 1.1. *Let $P, Q \in \mathcal{P}(H)$. The following are equivalent:*
(i) $P \leq Q$.
(ii) $PQ = P$.
(iii) $QP = P$.
(iv) $Q - P$ *is a projection.*

The above proposition motivates the definition of the ordering on projections in an arbitrary C*-algebra, namely $p \leq q$ if and only if $pq = p$.

Naively, it is natural to endow $\mathcal{P}(H)$ with the topology it inherits from the operator norm topology on $\mathcal{B}(H)$, however it is easy to see that this topology is not separable, and hence does not give $\mathcal{P}(H)$ the structure of a Polish space. Recall that the *strong operator topology* on $\mathcal{B}(H)$ is the topology induced by the family of seminorms $T \mapsto \|Tx\|$ for $x \in H$, or equivalently, the topology of pointwise convergence of operators on H. One can show:

Proposition 1.2. *$\mathcal{P}(H)$ is a Polish space when endowed with the strong operator topology.*

Consequently, when referring to Borel and analytic subsets of $\mathcal{P}(H)$, it will be with this topology in mind. It is an occasionally useful fact to know that the *weak operator topology* on $\mathcal{B}(H)$, namely the topology generated by the seminorms $T \mapsto |\langle Tx, y\rangle|$ for $x, y \in H$, coincides with the strong operator topology when restricted to $\mathcal{P}(H)$. However, $\mathcal{P}(H)$ is not compact in this topology. For the above proposition, and related issues, see §2.5 of [18].

Recall that an operator $T \in \mathcal{B}(H)$ is *compact* if the image of the closed unit ball $B \subset H$ under T is precompact. Equivalently, for operators on a Hilbert space, T is a limit of finite rank operators, or T is weak-norm continuous when restricted to B. Denote by $\mathcal{K}(H)$ the collection of all compact operators in H. The following is a well known theorem of operator theory (which requires H to be infinite dimensional and separable), see §4.1 of [23].

Theorem 1.1. *$\mathcal{K}(H)$ is the unique proper, closed (with respect to the operator norm topology), *-closed ideal in $\mathcal{B}(H)$.*

Definition 1.4. The *Calkin algebra* is the quotient C*-algebra $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$. Denote by $\pi : \mathcal{B}(H) \to \mathcal{C}(H)$ the quotient map.

The study of operators modulo $\mathcal{K}(H)$ (*modulo compact perturbation*) initially arose in the study of integral equations, via *Fredholm operators* (operators whose image in $\mathcal{C}(H)$ is invertible), see §3.3 of [24]. The program of classifying operators modulo compact perturbation was begun by Weyl and von Neumann, and culminated in the work of Brown-Douglas-Fillmore on classifying *essentially normal operators* up to unitary equivalence; see [2] and Ch. 6 of [3].

The set theoretic interest in the Calkin algebra originally derives from the following sequence of theorems, suggesting an analogy between the structure of $\mathcal{P}(\omega)/\mathrm{Fin}$ and $\mathcal{C}(H)$. Recall that the *Continuum Hypothesis* (CH) is the statement $2^{\aleph_0} = \aleph_1$. *Martin's Axiom* (MA), the *Open Coloring Axiom* (OCA), and the *Proper Forcing Axiom* (PFA) are alternate axioms of set theory known to be unprovable, but consistent with the usual ZFC axioms (PFA requires the consistency of certain *large cardinal axioms*), and PFA implies both OCA and MA. See [22] for a detailed survey of these axioms.

Theorem 1.2 (W. Rudin [27]). (CH) *There are $2^{2^{\aleph_0}}$ many automorphisms of the boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$.*

Theorem 1.3 (Shelah-Steprans [29], Velickovic [37]). (OCA + MA) *Every automorphism of $\mathcal{P}(\omega)/\mathrm{Fin}$ is induced by a bijection $\omega \setminus n \to \omega \setminus m$, for some $m, n \in \omega$; in particular, there are only $2^{\aleph_0}$ many such automorphisms. (Such automorphisms are said to be* trivial*.)*

Theorem 1.4 (Philips-Weaver [25]). (CH) *There are $2^{2^{\aleph_0}}$ many automorphisms of the C*-algebra $\mathcal{C}(H)$.*

Theorem 1.5 (Farah [11]). (OCA) *Every automorphism of $\mathcal{C}(H)$ is induced by a unitary operator on H acting by conjugation; in particular, there are only $2^{\aleph_0}$ many such automorphisms. (Such automorphisms are said to be* inner*.)*

Due to its order-theoretic resemblance with $\mathcal{P}(\omega)/\mathrm{Fin}$ (as we will see below), we will focus our attention on the set of projections in $\mathcal{C}(H)$. Observe that since the map π is a *-homomorphism, π maps projections on H to projections in $\mathcal{C}(H)$, the set of which we denote by $\mathcal{P}(\mathcal{C}(H))$. In fact, using some spectral theory, one can show that every projection in $\mathcal{P}(\mathcal{C}(H))$ occurs in this way. A proof can be found in [38].

Proposition 1.3. *If $p \in \mathcal{P}(\mathcal{C}(H))$, then there is a projection $P \in \mathcal{P}(H)$ such that $\pi(P) = p$.*

$\mathcal{P}(\mathcal{C}(H))$ is partially ordered by the relation $p \leq q$ if and only if $pq = p$, with $p < q$ if $p \leq q$ and $p \neq q$. As before, we wish to avoid studying equivalence classes explicitly, so we can study $\mathcal{P}(\mathcal{C}(H))$ by pulling back the aforementioned ordering to $\mathcal{P}(H)$. The previous proposition tells us that this is equivalent to studying $\mathcal{P}(\mathcal{C}(H))$. So, for $P, Q \in \mathcal{P}(H)$ we define

$$P <_{\mathrm{ess}} Q \quad \text{if and only if} \quad \pi(P) < \pi(Q),$$

and likewise say $P \leq_{\mathrm{ess}} Q$ if $\pi(P) \leq \pi(Q)$ (P is *essentially below* Q), and $P \equiv_{\mathrm{ess}} Q$ if $P \leq_{\mathrm{ess}} Q$ and $Q \leq_{\mathrm{ess}} P$ (P is *essentially equivalent* to Q). If PQ is compact (i.e., $\pi(PQ) = 0$), we say that P and Q are *essentially orthogonal*. As is the case for $\leq$, we have:

Proposition 1.4. *Let $P, Q \in \mathcal{P}(H)$. The following are equivalent:*
(i) $P \leq_{\mathrm{ess}} Q$.
(ii) $PQ \equiv_{\mathrm{ess}} P$ *(i.e., $P(I - Q)$ is compact).*
(iii) $QP \equiv_{\mathrm{ess}} P$ *(i.e., $(I - Q)P$ is compact).*

The following informative fact is not hard to verify:

Proposition 1.5. *$<_{\text{ess}}$, $\leq_{\text{ess}}$ and $\equiv_{\text{ess}}$ are all Borel relations on $\mathcal{P}(H)$, that is, they are Borel as subsets of $\mathcal{P}(H) \times \mathcal{P}(H)$, when $\mathcal{P}(H)$ has the strong operator topology.*

The remainder of this talk will be devoted to comparing and contrasting the structure of $(\mathcal{P}(H), <_{\text{ess}})$ with that of $(\mathcal{P}(\omega), \subset^*)$. The study of the order-theoretic properties of $(\mathcal{P}(H), <_{\text{ess}})$ was begun by Hadwin in [12], and further extended by Wofsey [39] and Zamora-Aviles [40] [41].

We should note here that the relation $<_{\text{ess}}$ is strictly weaker than the relation $<_f$ given by $P <_f Q$ if and only if $\text{ran}(P)$ is contained in a finite dimensional extension of $\text{ran}(Q)$. This is seen through the following example.

Example 1.1. Fix an orthonormal basis $(e_n)_{n\in\omega}$ for H. Define

$$\begin{aligned} P &= \text{the projection onto } \overline{\text{span}}\{e_{2n} + \frac{1}{n}e_{2n+1} : n \in \omega\} \\ Q &= \text{the projection onto } \overline{\text{span}}\{e_{2n} : n \in \omega\} \end{aligned}$$

It is easy to check that $P \wedge Q = 0$, so they are incomparable with respect to $\leq$, and $\leq_f$. We claim that $Q - P$ is compact, and hence $P \equiv_{\text{ess}} Q$. Given $x = \sum_{n=0}^{\infty} \alpha_n e_n \in H$ arbitrary, one can compute that

$$(Q-P)x = \sum_{n=0}^{\infty} \alpha_{2n} \left(1 - \frac{n^2}{n^2+1}\right) e_{2n} - \sum_{n=0}^{\infty} \alpha_{2n+1} \frac{1}{n^2+1} e_{2n+1} - \sum_{n=0}^{\infty} \frac{\alpha_{2n+1}}{n+\frac{1}{n}} e_{2n} - \sum_{n=0}^{\infty} \frac{\alpha_{2n}}{n+\frac{1}{n}} e_{2n+1},$$

from which it is easy to verify that $P - Q$ is compact. This gives a prototypical example of distinct projections P and Q with $P \equiv_{\text{ess}} Q$.

An explicit connection between $(\mathcal{P}(\omega), <^*)$ and $(\mathcal{P}(H), <_{\text{ess}})$ can be made via the following construction: Fix an orthonormal basis $E = \{e_n : n \in \omega\}$ for H. For each $x \subseteq \omega$, let

$$P_x^E = \text{the projection onto } \overline{\text{span}}\{e_n : n \in x\}.$$

The map $\mathcal{P}(\omega) \to \mathcal{P}(H) : x \mapsto P_x^E$ is called the *diagonal embedding* (with respect to E). One can easily check that the range of the diagonal embedding is a commutative family of *simultaneously diagonalized* projections, that is, by identifying $H = \ell^2(\omega)$ and E with the standard basis of $\ell^2(\omega)$, the projection P_x^E corresponds exactly to the multiplication operator $M_x \in \ell^\infty(\omega)$ acting on H, where x is thought of as its own characteristic function. In the terminology of C*-algebras, the range of the diagonal embedding is contained in an atomic (maximal) abelian self-adjoint subalgebra (an *atomic masa*). It is trivial to verify that the diagonal embedding satisfies:

Proposition 1.6. *For $x, y \subseteq \omega$,*
(i) *$x \subset y$ if and only if $P_x^E < P_y^E$.*
(ii) *$x \cap y = \emptyset$ if and only if $P_x^E P_y^E = 0$.*
(iii) *$P_{x\cap y}^E = P_x^E \wedge P_y^E = P_x^E P_y^E$.*
(iv) *$P_{x\cup y}^E = P_x^E \vee P_y^E$.*
(v) *$P_{\omega\setminus x}^E = (P_x^E)^\perp = I - P_x^E$.*

In fact, more is true; the diagonal embedding is a *reduction* of the relation $\subset^*$ to $<_{\text{ess}}$:

Proposition 1.7. *For $x, y \subseteq \omega$,*
(i) *$x \subset^* y$ if and only if $P_x^E <_{\text{ess}} P_y^E$.*
(ii) *$x \cap y$ is finite if and only if $P_x^E P_y^E$ is compact (in fact, finite rank).*

One can also show:

Proposition 1.8. *The diagonal embedding is continuous when $\mathcal{P}(H)$ is endowed with the strong operator topology.*

In the image of the diagonal embedding, essentially equivalence and equivalence modulo finite dimensional extensions coincide. This, combined with the fact that its range is a commutative family, suggests that the diagonal embedding is only capturing a small portion of the phenomena in $(\mathcal{P}(H), <_{\text{ess}})$.

The following lemma, due to Farah [10], shows that the diagonal embedding can capture *countable* essentially commuting families in $\mathcal{P}(H)$, up to essential equivalence.

Lemma 1.1 (Farah). *If $\{P_n : n < \omega\}$ is a sequence of essentially commuting projections, i.e., $P_n P_m \equiv_{\text{ess}} P_m P_n$ for all $m, n < \omega$, then there is an orthonormal basis $E = \{e_n : n < \omega\}$ of H and sets $x_n \subseteq \omega$ for which $P_n \equiv_{\text{ess}} P_{x_n}^E$ for all $n < \omega$.*

Note that the previous lemma is false even for commuting families of projections if we require $P_n = P_{x_n}^E$. (Take $H = L^2([0,1])$, and let P_n be the multiplication operator corresponding the characteristic function of the nth rational subinterval, in some enumeration. These projections have no common eigenvector, and thus cannot be simultaneously diagonalized in this fashion.) Farah has also shown that the lemma as stated is false for essentially commuting families of size $\aleph_1$, see §5 of [10].

2 Gaps

2.1 Gaps in $(\mathcal{P}(\omega), \subset^*)$

The presence of *gaps* in a partial order gives an indication of the failure of the order to be *complete* or *saturated*. For example, in $(\mathcal{P}(\omega), \subset)$ (not modulo finite), whenever we have a pair (A, B) with $A, B \subseteq \mathcal{P}(\omega)$, and $a \subseteq b$ for all $a \in A$ and $b \in B$, there is a $c \subseteq \omega$ such that $a \subseteq c$ and $c \subseteq b$ for all $a \in A$ and $b \in B$, namely $c = \bigcup A$. Likewise, if A and B are sets of real numbers such that $a \leq b$ for all $a \in A$ and $b \in B$, in the usual ordering on $(\mathbb{R}, <)$, one can just take $c = \sup A$. The ordering $(\mathcal{P}(\omega), \subset^*)$, however, fails to be complete in dramatic fashion.

Definition 2.1. (a) A *pregap* in $(\mathcal{P}(\omega), \subset^*)$ is a pair (A, B) with $A, B \subseteq \mathcal{P}(\omega)$ such that $a \subseteq^* b$ for all $a \in A$ and $b \in B$.
(b) If (A, B) is pregap and $c \subseteq \omega$ is such that $a \subseteq^* c$ and $c \subseteq^* b$ for all $a \in A$ and $b \in B$, then c is said to *interpolate* (or *split*) (A, B).
(c) If (A, B) is a pregap such that no $c \subseteq \omega$ interpolates it, then we say that (A, B) is a *gap*.

A diagonalization argument shows that pregaps with *countable* sides can always be interpolated.

Proposition 2.1. *If (A, B) is a pregap in $(\mathcal{P}(\omega), \subset^*)$ with A and B countable, then (A, B) can be interpolated.*

The easiest gaps to construct are those originally described by Luzin [19].

Definition 2.2. A *Luzin gap* in $(\mathcal{P}(\omega), \subset^*)$ is a pregap $(\{a_i\}_{i\in I}, \{b_i\}_{i\in I})$ such that I is uncountable, and
(i) for all $i \in I$, $a_i \subset b_i$, and
(ii) for all $i \neq j$ in I, one of $a_i \setminus b_j$ or $a_j \setminus b_i$ is nonempty.

It is fairly easy to verify the following:

Lemma 2.1. *A Luzin gap is a gap.*

Example 2.1. We will build the a Luzin gap in $(2^{<\omega}, \subset^*)$ rather than $(\mathcal{P}(\omega), \subset^*)$; any bijection between $2^{<\omega}$ will ω provide the desired gap. For each $x \in 2^\omega$, let

$$a_x = \{\sigma \in 2^{<\omega} : \sigma\char`\^0 \sqsubseteq x\},$$

and

$$b_x = 2^{<\omega} \setminus \{\sigma \in 2^{<\omega} : \sigma\char`\^1 \sqsubseteq x\}.$$

Clearly, for each $x \in 2^\omega$, $a_x \subset b_x$. For $x \neq y$ in 2^ω, let σ be the maximal initial segment of both x and y, then either $\sigma \in a_x \setminus b_y$ or $\sigma \in a_y \setminus b_x$. Thus, $(\{a_x\}_{x\in 2^\omega}, \{b_y\}_{y\in 2^\omega})$ is a Luzin gap. Moreover, one can easily verify that the maps $x \mapsto a_x$ and $y \mapsto b_y$ are continuous maps $2^\omega \to 2^\omega$, the sides of the gap are Borel, and that this is preserved by moving from $2^{<\omega}$ to ω.

Definition 2.3. (a) A (pre)gap (A, B) in $(\mathcal{P}(\omega), \subset^*)$ is *linear* if A is a $\subset^*$-increasing chain, and B is a $\subset^*$-decreasing chain.

(b) A (κ, λ)-*gap*, for κ and λ regular cardinals, is a linear gap (A, B) where A has cofinality κ and B has coinitiality λ^*, the reverse of λ. (Without loss of generality, we restrict our attention to (κ, λ)-gaps (A, B) in which A has order type κ and B has order type λ^*.)

It is easy to check that a (κ, λ)-gap exists if and only if a (λ, κ)-gap exists. We call the set of pairs of regular cardinals (κ, λ) such that a (κ, λ)-gap exists in $(\mathcal{P}(\omega), \subset^*)$ the *linear gap spectrum* of $(\mathcal{P}(\omega), \subset^*)$.

Example 2.2. A simple Zorn's Lemma argument produces an (κ, ω)-gap in $(\mathcal{P}(\omega), \subset^*)$, for some uncountable κ (such gaps are called *Rothberger gaps*, for [26]): Let $\{b_j\}_{j\in\omega}$ be a such that each $b_j \subset \omega$, and $b_{j+1} \subset^* b_j$. Define a set $\mathcal{G}$ by $A \in \mathcal{G}$ if and only if $A \subset \mathcal{P}(\omega)$ is well-ordered by $\subset^*$, and if $a \in A$, then a is infinite and for all $j \in \omega$, $a \subset^* b_j$. $\mathcal{G} = \emptyset$ by the non-existence of $(0, \omega)$-gaps. Order $\mathcal{G}$ by $A \prec B$ if and only if $A \subsetneq B$ and for all $a \in A$, $b \in B \setminus A$, $a \subset^* b$ with $b \setminus a$ infinite. It is clear that $\mathcal{G}$ is closed under taking unions of $\prec$-chains. By Zorn's Lemma, $\mathcal{G}$ has a $\prec$-maximal element, which has a cofinal subset of minimal (regular) order type κ, say $\{a_\alpha\}_{\alpha<\kappa}$. By maximality, $(\{a_\alpha\}_{\alpha<\kappa}, \{b_j\}_{j\in\omega})$ forms a (κ, ω)-gap.

Definition 2.4. A gap (A, B) in $(\mathcal{P}(\omega), \subset^*)$ is a *Hausdorff gap* if A is σ-directed by $\subseteq^*$, and B is reverse σ-directed by $\subseteq^*$.

Note that any (κ, κ)-gap for κ uncountable (regular) must be Hausdorff. The following classical theorem of Hausdorff establishes the existence of such gaps. A clear proof can be found in [17].

Theorem 2.1 (Hausdorff [14] [15])**.** *There exists an (ω_1, ω_1)-gap in $(\mathcal{P}(\omega), \subset^*)$.*

Observe that if κ is the minimum cardinal such that a (κ, κ)-gap exists in $(\mathcal{P}(\omega), \subset^*)$, then this theorem shows that $\kappa = \aleph_1$, a rare instance of a *cardinal characteristic of the continuum* being computed in ZFC. Hausdorff's example is a remarkably versatile tool in studying the combinatorial properties of the reals; it can be used to produce a decreasing sequence of $\aleph_1$ many uncountable F_σ subsets of 2^ω with empty intersection, a partition of $\mathbb{R}$ into $\aleph_1$ many disjoint Borel sets, and an uncountable universal measure zero subset of $\mathbb{R}$. One can show that Hausdorff's example is equivalent (in an appropriate sense) to a *special* gap, and Kunen has shown that such gaps cannot be destroyed by any ω_1-preserving (in particular, proper) forcing. Thus, in a sense, these gaps are the only ZFC examples of (κ, κ)-gaps. See [28] for details on these, and other matters related to forcing and gaps. The situation was further clarified by the following theorem of Todorcevic. Note that the hypotheses OCA and $2^{\aleph_0} = \aleph_2$ are both consequences of PFA, see [31] and [22].

Theorem 2.2 (Todorcevic [31])**.** $(\mathsf{OCA} + (2^{\aleph_0} = \aleph_2))$ *The linear gap spectrum of $(\mathcal{P}(\omega), \subset^*)$ is exactly (ω, ω_2), (ω_2, ω), and (ω_1, ω_1).*

We have seen that (nonlinear) gaps (A, B) in $(\mathcal{P}(\omega), \subset^*)$, such as the Luzin-type example above, can be constructed with A and B both Borel. The following key theorem of Todorcevic shows Hausdorff gaps cannot be constructed with this low level of complexity.

Theorem 2.3 (*Analytic Gaps Theorem*, Todorcevic [32])**.** *If (A, B) is a pregap in $(\mathcal{P}(\omega), \subset^*)$ such that A is σ-directed by $\subseteq^*$, B is reverse σ-directed by $\subseteq^*$, and one of A or B is analytic (as a subset of $\mathcal{P}(\omega)$), then (A, B) can be interpolated.*

This theorem can be summarized by saying that "analytic Hausdorff gaps do not exist" in $(\mathcal{P}(\omega), \subset^*)$. In particular, if (A, B) is such a gap, A cannot be an analytic P-ideal, nor can $\{c \subseteq \omega : \omega \setminus c \in B\}$.

The analysis of gaps in $(\mathcal{P}(\omega), \subset^*)$, and consequences of PFA, has arisen naturally in several situations, including:

(1) The independence of *Kaplansky's Conjecture*: Every homomorphism from $C(X)$ (for compact Hausdorff X) into a Banach algebra is continuous. [4]

(2) The theory of *strong homology* for certain topological spaces. [5]

(3) The study of compact subsets of the space of *Baire class 1* functions, i.e., those functions which are pointwise limits of continuous functions. [35]

(4) The *metrizability problem* for Fréchet groups: Is every separable Fréchet group metrizable? [36]

(5) The *separable quotients problem* for Banach spaces: Does every infinite dimensional Banach space have an infinite dimensional *separable* quotient? [1]

2.2 Gaps in $(\mathcal{P}(H), <_{\rm ess})$

We now turn to studying gaps in $(\mathcal{P}(H), <_{\rm ess})$. The relevant definitions are analogous to those in $(\mathcal{P}(\omega), \subset^*)$.

Definition 2.5. (a) A *pregap* in $(\mathcal{P}(H), <_{\rm ess})$ is a pair (A, B) with $A, B \subseteq \mathcal{P}(H)$ such that $P \leq_{\rm ess} Q$ for all $P \in A$ and $Q \in B$.

(b) If (A, B) is pregap in $(\mathcal{P}(H), <_{\rm ess})$ and $R \in \mathcal{P}(H)$ is such that $P \leq_{\rm ess} R$ and $R \leq_{\rm ess} Q$ for all $P \in A$ and $Q \in B$, then R is said to *interpolate* (or *split*) (A, B).

(c) If (A, B) is a pregap in $(\mathcal{P}(H), <_{\rm ess})$ such that no $R \in \mathcal{P}(H)$ interpolates it, then we say that (A, B) is a *gap*.

(d) A (pre)gap (A, B) is *linear* if A is a $<_{\rm ess}$-increasing chain, and B is a $<_{\rm ess}$-decreasing chain.

(e) A (κ, λ)-*gap*, for κ and λ regular cardinals, is a linear gap (A, B) where A has cofinality κ and B has coinitiality λ^*. (Again, we restrict our attention to (κ, λ)-gaps (A, B) in which A has order type κ and B has order type λ^*.)

(f) A gap (A, B) is a *Hausdorff gap* if A is σ-directed by $<_{\rm ess}$, and B is reverse σ-directed by $<_{\rm ess}$.

Proposition 2.2. *If (A, B) is a pregap in $(\mathcal{P}(H), <_{\rm ess})$ with A and B countable essentially commuting families (such as $<^*$-chains), then (A, B) can be interpolated.*

Proof: Suppose that $A = \{P_i : i \in \omega\}$ and $B = \{Q_j : j \in \omega\}$. Note that if $P \in A$ and $Q \in B$, then P and Q essentially commute, because they are $<^*$-comparable. Using a previous lemma, there is an orthonormal basis $E = \{e_n : n \in \omega\}$ for H and sets $x_i, y_j \subset \omega$, for $i, j \in \omega$, such that $\pi(P_i) = \pi(P^E_{x_i})$ and $\pi(Q_j) = \pi(P^E_{y_j})$ for $i, j \in \omega$. Then, $(\{x_i : i \in \omega\}, \{y_j : j \in \omega\})$ is pregap with countable sides in $(\mathcal{P}(\omega), \subset^*)$, and is thus interpolated by some $c \subseteq \omega$. It follows that P^E_c interpolates (A, B). Q.E.D.

Since the diagonal embedding is a homomorphism of $\subset^*$ to $<_{\rm ess}$, it sends pregaps in $(\mathcal{P}(\omega), \subset^*)$ to pregaps in $(\mathcal{P}(H), <_{\rm ess})$. We will see below that much more is true. To this end, we will need to develop a key piece of machinery for constructing P-ideals in $\mathcal{P}(\omega)$.

Definition 2.6. A *submeasure* on ω is a map $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ satisfying
(i) $\varphi(\emptyset) = 0$,
(ii) $\varphi(x) \leq \varphi(y)$ whenever $x \subseteq y$,
(iii) $\varphi(x \cup y) \leq \varphi(x) + \varphi(y)$ for all $x, y \subseteq \omega$, and
(iv) $\varphi(\{n\}) < \infty$ for all $n \in \omega$.
φ is *lower semincontinuous* (lsc) if $\varphi(x) = \lim_{n\to\infty} \varphi(x \cap n)$ for all $x \subseteq \omega$.

Definition 2.7. Let φ be a lsc submeasure on ω. The *exhaustive ideal* of φ is

$$\mathrm{Exh}(\varphi) = \{x \subseteq \omega : \lim_{n\to\infty} \varphi(x \setminus n) = 0\}.$$

A simple example of a lsc submeasure φ on ω is given by counting measure, in which case $\mathrm{Exh}(\varphi) = \mathrm{Fin}$.

Proposition 2.3 (Folklore)**.** *Let φ be a lsc submeasure on ω. Then* $\mathrm{Exh}(\varphi)$ *is an analytic (in fact, $F_{\sigma\delta}$) P-ideal in $\mathcal{P}(\omega)$.*

We note that a beautiful theorem of Solecki [30] shows that *every* analytic P-ideal in $\mathcal{P}(\omega)$ arises in this fashion.

Returning to the setting at hand, fix an orthonormal basis $E = \{e_n : n \in \omega\}$ for our Hilbert space H. For each $P \in \mathcal{P}(H)$, define

$$\mathcal{I}_P = \{x \subseteq \omega : PP_x^E \text{ is compact}\}.$$

If P is an infinite rank projection, it is easy to see that $\mathcal{I}_P$ is a proper, nonprinciple ideal on ω. In fact:

Lemma 2.2 (Steprans)**.** *For P an infinite rank projection, $\mathcal{I}_P$ is an analytic P-ideal on ω.*

Proof: The idea behind the proof is to define $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ by

$$\varphi(x) = \|PP_x^E\| = \|P_x^E P\|.$$

Then, one shows that φ is a lsc submeasure, and that $\mathcal{I}_P = \mathrm{Exh}(\varphi)$. (Details omitted.) Q.E.D.

Theorem 2.4 (Zamora-Aviles [40] [41])**.** *Given an orthonormal basis E for H, the diagonal embedding $\mathcal{P}(\omega) \to \mathcal{P}(H) : x \mapsto P_x^E$ is gap preserving.*

Proof: This proof is based on that of a result in [33]. Let (A, B) be a gap in $(\mathcal{P}(\omega), \subset^*)$, and let A^* and B^* be the images of A and B in $\mathcal{P}(H)$ under the diagonal embedding. (A^*, B^*) is a pregap in $(\mathcal{P}(H), <_{\mathrm{ess}})$. We claim that it is in fact a gap. Suppose not, then there is an infinite rank $P \in \mathcal{P}(H)$ which interpolates (A^*, B^*). That is, $P_a^E(I - P)$ is compact for every $a \in A$, and $P(I - P_b^E)$ is compact for every $b \in B$. Let $\mathcal{I}_P$ and $\mathcal{I}_{I-P}$ be the ideals on ω associated to P and $I - P$ as defined above. As we have seen, both are analytic P-ideals. Let $\mathcal{F}_P = \{x \subseteq \omega : \omega \setminus x \in \mathcal{I}_P\}$. Then, $A \subseteq \mathcal{I}_{I-P}$ and $B \subseteq \mathcal{F}_P$. We claim that $(\mathcal{I}_{I-P}, \mathcal{F}_P)$ forms a pregap, or equivalently, for every $a \in \mathcal{I}_{I-P}$ and $b \in \mathcal{I}_P$, $a \cap b \in \mathrm{Fin}$. For such an a and b,

$$P_{a\cap b}^E = P_{a\cap b}^E(I - P) + P_{a\cap b}^E P = P_b^E P_a^E(I - P) + P_a^E P_b^E P \leq P_a^E(I - P) + P_b^E P,$$

but the latter is compact, so $a \cap b \in \mathrm{Fin}$ as claimed. Thus, by the analytic gaps theorem, $(\mathcal{I}_{I-P}, \mathcal{F}_P)$ can be interpolated, but such an interplant would also interpolate (A, B), contradicting the latter being a gap. Thus, (A^*, B^*) is a gap. Q.E.D.

Consequently, all of the gap phenomena occurring in $(\mathcal{P}(\omega), \subset^*)$ also occurs in $(\mathcal{P}(H), <_{\mathrm{ess}})$. In fact, the gap structure of $(\mathcal{P}(H), <_{\mathrm{ess}})$ is even richer.

Theorem 2.5 (Zamora-Aviles [40] [41])**.** *There is an analytic Hausdorff gap in $(\mathcal{P}(H), <_{\mathrm{ess}})$.*

Proof: Fix a sequence $\{J_n : n \in \omega\}$ of consecutive (i.e., $\max(J_n) < \min(J_{n+1})$), disjoint intervals in ω with $|J_n| = 2^{2^n}$ for $n \in \omega$. We will build a sequence of finite dimensional Hilbert spaces $\{H_n : n \in \omega\}$ such that each H_n has two orthonormal bases $\{e_i : i \in J_n\}$ and $\{f_i : i \in J_n\}$ such that

$$\langle e_i, f_j \rangle^2 = \frac{1}{2^{2^n}}, \text{ for } i, j \in J_n.$$

We will let $H = \bigoplus_{n\in\omega} H_n$ (the Hilbert space direct sum, c.f., 3.1.5 in [24]), and this is the Hilbert space on which we build the gap. Note that in this case, if $v \in H_n$ and $w \in H_m$ for $n \neq m$, then $\langle v, w\rangle = 0$. We construct the H_n by recursion. Let H_0 be the (2-dimensional) Hilbert space generated by orthonormal bases $\{(1,0),(0,1)\}$ and $\{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$. It is clear that, by indexing these basis vectors by J_0, this space satisfies the desired property. Suppose that we have constructed H_n as desired. Let $H_{n+1} = H_n \otimes H_n$, the Hilbert space tensor product. Then, $\{e_i \otimes e_j : i, j \in J_n\}$ and $\{f_i \otimes f_j : i, j \in J_n\}$ are orthonormal bases of H_{n+1} and

$$\langle e_i \otimes e_j, f_k \otimes f_l\rangle^2 = \langle e_i, f_k\rangle^2 \langle e_j, f_l\rangle^2 = \frac{1}{2^{2^n 2}} = \frac{1}{2^{2^{n+1}}}$$

as desired. By indexing these orthonormal bases by J_{n+1}, we have satisfied the desired property.

By construction, H has a pair of orthonormal bases $E = \{e_i : i \in \omega\}$ and $F = \{f_i : i \in \omega\}$ where $\{e_i : i \in J_n\}$ and $\{f_i : i \in J_n\}$ are the orthonormal bases of H_n described above.

Define $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ by

$$\varphi(x) = \sup\left\{\frac{|x \cap J_n|}{n} : n \in \omega\right\}, \text{ for } x \subset \omega.$$

We claim that φ is a lsc submeasure on ω. Clearly, $\varphi(\emptyset) = \varphi(\{m\}) < \infty$ for $m \in \omega$, and $\varphi(x) \leq \varphi(y)$ for $x \subset y \subset \omega$. Likewise, $\varphi(x \cup y) \leq \varphi(x) + \varphi(y)$ for $x, y \subset \omega$. To see that φ is lsc, we consider two cases. If $\varphi(x) = \infty$, then for each $K \in \omega$, there is an n_K such that

$$\frac{|x \cap J_{n_K}|}{n_K} > K.$$

Given K, if $m > \max(J_{n_K})$, then

$$\frac{|(x \cap m) \cap J_{n_K}|}{n_K} = \frac{|x \cap J_{n_K}|}{n_K} > K,$$

and so $\varphi(x \cap m) > K$, showing that $\lim_m \varphi(x \cap m) = \infty = \varphi(x)$. If $\varphi(x) < \infty$, then for every $\epsilon > 0$, there is an n_ϵ such that

$$0 \leq \varphi(x) - \frac{|x \cap J_{n_\epsilon}|}{n_\epsilon} < \epsilon.$$

Given $\epsilon > 0$, if $m > \max\{J_{n_\epsilon}\}$, then

$$0 \leq \varphi(x \cap m) - \frac{|(x \cap m) \cap J_{n_\epsilon}|}{n_\epsilon} = \varphi(x) - \frac{|x \cap J_{n_\epsilon}|}{n_\epsilon} < \epsilon,$$

and so $\lim_m \varphi(x \cap m) = \varphi(x)$. Thus, φ is lsc. In particular, $\mathrm{Exh}(\varphi)$ is an analytic ($F_{\sigma\delta}$) P-ideal. Note that $\mathrm{Exh}(\varphi)$ contains infinite elements, such as any set $x \subseteq \omega$ satisfying $|x \cap J_n| = 1$ for all n.

Let
$$A = \{P_x^E : x \in \mathrm{Exh}(\varphi)\} \quad \text{and} \quad B = \{P_x^F : x \in \mathrm{Exh}(\varphi)\}.$$
Since $\mathrm{Exh}(\varphi)$ is an analytic P-ideal, both A and B are σ-directed analytic sets, as the diagonal embedding is order preserving and continuous. We claim that:

(1) For all $P_x^E \in A$ and $P_y^F \in B$, $P_y^E P_x^F$ is compact (i.e., $P_x^E$and P_y^F are essentially orthogonal), and

(2) there does not exists a projection $P \in \mathcal{P}(H)$ such that $(I-P)P_x^E$ and PP_y^F are compact, for all $P_x^E \in A$ and $P_y^F \in B$.

Taken together, these imply that $(A, \{I - P_y^F : y \in \mathrm{Exh}(\varphi)\})$ forms a gap.

For $x \subset \omega$, we denote by $x_n = x \cap J_n$. To verify (1), let $P_x^E \in A$ and $P_y^F \in B$. Let $u \in \mathrm{ran}(P_x^E)$, say with $u = \sum_{i\in x} a_i e_i$. Then
$$P_y^F P_x^E u = P_y^F u = \sum_{j\in y}\langle u, f_j\rangle f_j = \sum_{j\in y}\left\langle \sum_{i\in x} a_i e_i, f_j\right\rangle f_j = \sum_{j\in y}\left(\sum_{i\in x} a_i\langle e_i, f_j\rangle\right) f_j$$
$$= \sum_{n\in\omega}\left(\sum_{j\in y_n}\left(\sum_{i\in x_n} a_i\langle e_i, f_j\rangle\right) f_j\right).$$
In order to show that $P_y^F P_x^E$ is compact, or equivalently, weak-norm continuous when restricted to the unit ball of H, it suffices to show that if $\|u\| \le 1$, then $\sum_{j\in y_n}\left(\sum_{i\in x_n} a_i\langle e_i, f_j\rangle\right) f_j$ is (summably) small in norm for large n. Observe
$$\left\|\sum_{j\in y_n}\left(\sum_{i\in x_n} a_i\langle e_i, f_j\rangle\right) f_j\right\|^2 = \sum_{j\in y_n}\left|\sum_{i\in x_n} a_i\langle e_i, f_j\rangle\right|^2 \le \frac{1}{2^{2^n}}\sum_{j\in y_n}\left(\sum_{i\in x_n}|a_i|\right)^2 \le \frac{1}{2^{2^n}}|x_n||y_n|^2.$$
But, $x, y \in \mathrm{Exh}(\varphi)$, so there is an N such that for $n > N$, $|x_n| \le n$ and $|y_n| \le n$, in which case,
$$\left\|\sum_{j\in y_n}\left(\sum_{i\in x_n} a_i\langle e_i, f_j\rangle\right) f_j\right\|^2 \le \frac{n^3}{2^{2^n}}$$
as desired. It remains to prove (2). Suppose not, so there is an infinite rank projection $P \in \mathcal{P}(H)$ such that $(I-P)P_x^E$ and PP_x^F are compact for all $x \in \mathrm{Exh}(\varphi)$. Moreover, we claim that
$$\lim_{i,j}\langle (I-P)e_i, e_j\rangle = 0 \quad \text{and} \quad \lim_{i,j}\langle Pf_i, f_j\rangle = 0.$$
Consider the latter limit: If not, then there is an $\epsilon > 0$ and a cofinal sequence $\{n_i, \ell_j\}_{i,j\in\omega}$ with $\langle Pf_{n_i}, f_{\ell_j}\rangle \ge \epsilon$. By thinning down the aforementioned sequence, we can assume that $x = \{n_i : i \in \omega\} \in \mathrm{Exh}(\varphi)$, but then $\langle Pf_{n_i}, f_{\ell_j}\rangle \ge \epsilon$ witnesses that PP_x^F is not compact, since compact operators are weak-norm continuous on the unit ball. Likewise for the other limit.

Fix ϵ with $0 < \epsilon < 1$. By the above, we can find n large enough so that $\langle Pf_i, f_i\rangle < \epsilon$ and $\langle Pe_i, e_i\rangle > 1-\epsilon$ for $i \in J_n$. But then, if P_{H_n} is the projection onto H_n, we have that
$$\mathrm{tr}(P_{H_n} P P_{H_n}) < 2^{2^n}\epsilon,$$
using the basis $\{f_i : i \in J_n\}$, while
$$\mathrm{tr}(P_{H_n} P P_{H_n}) > 2^{2^n}(1-\epsilon),$$

using the basis $\{e_i : i \in J_n\}$. But trace is independent of basis, so this is a contradiction. Hence, no such P exists. Q.E.D.

It can also be shown, using the above techniques, that the linear gap spectrum of $(\mathcal{P}(H), <_{\text{ess}})$ can be strictly larger than that of $(\mathcal{P}(\omega), \subset^*)$.

Theorem 2.6 (Zamora-Aviles [40] [41]). (MA) *There is an $(2^{\aleph_0}, 2^{\aleph_0})$-gap in $(\mathcal{P}(H), <_{\text{ess}})$.*

Corollary 2.1. (OCA+MA+$(2^{\aleph_0} = \aleph_2)$) *The linear gap spectrum of $(\mathcal{P}(H), <_{\text{ess}})$ is strictly larger than that of $(\mathcal{P}(\omega), \subset^*)$.*

3 Questions

The following questions naturally arise from the investigations above. If unspecified, we intend these question to be answered in ZFC, or ZFC plus some collection of consequences of PFA (OCA, MA, $2^{\aleph_0} = \aleph_2$, etc).

Question 1. *Can we classify all analytic Hausdorff gaps in $(\mathcal{P}(H), <_{\text{ess}})$? More specifically, if (A, B) is analytic Hausdorff gap in $(\mathcal{P}(H), <_{\text{ess}})$, say with A and B (essentially) commutative families, then do A and B arise from a P-ideal on ω in the fashion of the gap constructed above?*

Question 2. *What is the linear gap spectrum of $(\mathcal{P}(H), <_{\text{ess}})$? In particular, is there an (ω_1, ω_2)-gap?*

Question 3. *Besides* Fin, *for which other nontrivial analytic (P)-ideals $\mathcal{I}$ in $\mathcal{P}(\omega)$, is there a reduction of $(\mathcal{P}(\omega), \subset_{\mathcal{I}})$ to $(\mathcal{P}(H), <_{\text{ess}})$? Can this reduction be made to preserves gaps?*

Particular candidates for this question are the *summable ideal*

$$\mathcal{I}_{\frac{1}{n}} = \{x \subset \omega : \sum_{n \in x} \frac{1}{n} < \infty\},$$

the *density zero ideal*

$$\mathcal{Z}_0 = \{x \subset \omega : \lim_n \frac{|x \cap n|}{n} = 0\},$$

and the F_σ ideal $\mathcal{I}$ described by Farah in §5.10 of [7] and Moore in [21]. It is know that over $\mathcal{I}_{\frac{1}{n}}$ and $\mathcal{I}$, there is an analytic Hausdorff gap (see [7] and [8] respectively), and thus gap preserving embeddings of the corresponding partial orders into $(\mathcal{P}(H), <_{\text{ess}})$ would explain the existence of such a gap in this structure. It remains open as to whether there is an analytic Hausdorff gap over $\mathcal{Z}_0$ (see [9]).

In [39], Wofsey begins to develop the theory of *maximal* essentially orthogonal (meo) families in $\mathcal{P}(H)$, in analogy to the theory of maximal almost disjoint (mad) families in $\mathcal{P}(\omega)$. It would be interesting and informative to further develop this theory. Of particular interest is the following question, inspired by a theorem of Mathias [20] which says that analytic mad families do not exist.

Question 4. *Are there analytic meo families in $\mathcal{P}(H)$?*

From a set theoretic perspective, whenever one investigates the combinatorial properties of a poset, it is natural to ask how (the positive elements of) that poset behaves as a notion of forcing. Recall that if $\mathcal{P}(\omega)^+ = \mathcal{P}(\omega) \setminus \mathrm{Fin}$, then $(\mathcal{P}(\omega)^+, \subset^*)$ is a σ-closed (hence proper and ω_1-preserving) forcing which adds a *Ramsey ultrafilter* on ω. Let $\mathcal{P}(H)^+ = \mathcal{P}(H) \setminus \mathcal{K}(H)$ (i.e., the infinite rank projections). It is easy to check, using Farah's lemma mentioned above, that $(\mathcal{P}(H)^+, <_{\mathrm{ess}})$ is also σ-closed.

Question 5. *What is $(\mathcal{P}(H)^+, <_{\mathrm{ess}})$ as a notion of forcing? What kind of objects does it add to the universe? Can we represent it as an iteration of other well-known forcings?*

We have mentioned that the diagonal embedding is a continuous reduction of $\subset^*$ to $<_{\mathrm{ess}}$, and thus also a continuous reduction of the *Borel equivalence relations* $\equiv^*$ to $\equiv_{\mathrm{ess}}$. Note that $\equiv^*$ is also known as E_0, and so this shows that $\equiv_{\mathrm{ess}}$ is not *smooth*, in the sense of [13]. We would like to further understand $\equiv_{\mathrm{ess}}$ as a Borel equivalence relation.

Question 6. *Within the hierarchies of Borel partial orders and equivalence relations under Borel reducibility, what is the relative complexity of $<_{\mathrm{ess}}$ and $\equiv_{\mathrm{ess}}$? Is the latter (above) a turbulent orbit equivalence relation (in the sense of [16])?*

Lastly, recalling that $\mathcal{P}(H)$ and the compact operators original emerge from operator theory, we ask:

Question 7. *Are there applications within operator theory/operator algebras to the study of gaps in $(\mathcal{P}(H), <_{\mathrm{ess}})$?*

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