Towards a selective (local) Gowers dichotomy

Iian Smythe

Cornell University

ASL North American Annual Meeting
University of Connecticut
May 25, 2016
Outline

1. Review of (local) Ramsey theory on $\omega$

2. Ramsey theory for block sequences in vector spaces

3. Local Ramsey theory for block sequences in vector spaces
Ramsey’s theorem

Theorem (Ramsey)

For every $n \in \omega$, if $A \subseteq [\omega]^n$ and $X \in [\omega]^{\omega}$, then there is a $Y \in [X]^{\omega}$ such that either $[Y]^n \cap A = \emptyset$ or $[Y]^n \subseteq A$.

- $[X]^n$ is the set of all $n$-element subsets of $X$, for $n \in \omega \cup \{\omega\}$.
- Good exercise in recursive constructions of length $2^{\aleph_0}$: The theorem is false if $n = \omega$. 
By putting definability restrictions on the partition, we obtain:

**Theorem (Silver)**

If $A \subseteq [\omega]^\omega$ is analytic and $X \in [\omega]^\omega$, then there is a $Y \in [X]^\omega$ such that either $[Y]^\omega \cap A = \emptyset$ or $[Y]^\omega \subseteq A$.

With more assumptions, we can go well beyond the analytic sets:

**Theorem (Shelah & Woodin)**

Assume $\exists$ supercompact $\kappa$. If $A \subseteq [\omega]^\omega$ is in $L(R)$ and $X \in [\omega]^\omega$, then there is a $Y \in [X]^\omega$ such that either $[Y]^\omega \cap A = \emptyset$ or $[Y]^\omega \subseteq A$. 

Iian Smythe (Cornell)
Local Ramsey theory

Theorem (Silver, Shelah & Woodin)

\[ \exists \text{ supercompact } \kappa \] If \( A \subseteq [\omega]^{\omega} \) is analytic (in \( L(\mathbb{R}) \)), then for any \( X \in [\omega]^{\omega} \), there is a \( Y \in [X]^{\omega} \) such that either \( [Y]^{\omega} \cap A = \emptyset \) or \( [Y]^{\omega} \subseteq A \).

Local Ramsey theory concerns “localizing” the witness \( Y \) above. That is, finding families \( \mathcal{H} \subseteq [\omega]^{\omega} \) such that, provided the given \( X \) is in \( \mathcal{H} \), \( Y \) can also be found in \( \mathcal{H} \).
Local Ramsey theory (cont’d)

Definition

- $\mathcal{H} \subseteq [\omega]^\omega$ is a coideal if it is the complement of a (non-trivial) ideal. Equivalently, it is a non-empty family such that
  - $X \in \mathcal{H}$ and $X \subseteq^* Y \implies Y \in \mathcal{H}$,
  - $X, Y \in [\omega]^\omega$ with $X \cup Y \in \mathcal{H} \implies X \in \mathcal{H}$ or $Y \in \mathcal{H}$.

- A coideal $\mathcal{H} \subseteq [\omega]^\omega$ is selective if whenever $X_0 \supseteq X_1 \supseteq \cdots$ are in $\mathcal{H}$, there is an $X \in \mathcal{H}$ such that $X/n \subseteq X_n$ for all $n \in X$.

Examples (of selective coideals)

- $[\omega]^\omega$
- $\mathcal{U}$ a selective (or sufficiently generic) ultrafilter
- $[\omega]^\omega \setminus \mathcal{I}$ where $\mathcal{I}$ is the ideal generated by an infinite a.d. family
Local Ramsey theory (cont’d)

Theorem (Mathias, Todorcevic)

(∃ supercompact κ.) Let \( H \subseteq [\omega]^{\omega} \) be a selective coideal. If \( A \subseteq [\omega]^{\omega} \) is analytic (in \( L(R) \)), then for any \( X \in H \), there is a \( Y \in H \upharpoonright X \) such that either \( [Y]^{\omega} \cap A = \emptyset \) or \( [Y]^{\omega} \subseteq A \).

Corollary

- (∃ supercompact κ.) If \( A \) is an infinite a.d. family which is analytic (in \( L(R) \)), then \( A \) fails to be maximal.
- (∃ supercompact κ.) A filter \( G \) is \( L(R) \)-generic for \( ([\omega]^{\omega}, \subseteq^*) \) if and only if \( G \) is selective.

As a result, selective ultrafilters are said to have “complete combinatorics” (see work of Blass, LaFlamme, Dobrinen)

An “abstract” version has recently been developed for topological Ramsey spaces (by Di Prisco, Mijares, & Nieto).
Block sequences in vector spaces

Let $E$ be an $\aleph_0$-dimensional vector space over $\mathbb{Q}$, with basis $(e_n)$.

**Definition**

- Given any vector $x = a_0e_0 + \cdots + a_ke_k$, its **support** (with respect to $(e_n)$) is $\text{supp}(x) = \{n : a_n \neq 0\}$.

- A **block sequence** (with respect to $(e_n)$) is a sequence $(x_n)$ of vectors such that $\max(\text{supp}(x_n)) < \min(\text{supp}(x_{n+1}))$, written $x_n < x_{n+1}$, for all $n$.

- For $X$ and $Y$ block sequences, if $X$ is block with respect to $Y$, we write $X \preceq Y$. Equivalently (for block sequences), $\langle X \rangle \subseteq \langle Y \rangle$.

Let $\mathbb{b}^\infty(E)$ be the (Polish) space of infinite block sequences in $E$.

- Abuse of terminology: “vectors” = non-zero vectors.

- $(\mathbb{b}^\infty(E), \preceq^*)$ (i.e., $\preceq$ modulo finite) is a $\sigma$-closed poset, equivalent to forcing with infinite dimensional subspaces of $E$. 
Ramsey theory for block sequences?

What would a Ramsey theorem block sequences in $E$ look like?

A “pigeonhole principle”: If $A \subseteq E$, there is an $X \in \mathbb{b}b^\infty(E)$ all of whose $\infty$-dimensional (block) subspaces are contained in one of $A$ or $A^c$.

Example

This is false. Let $A$ be vectors whose first coefficient, with respect to the basis $(e_n)$, is positive. There is no $X$ with the above property.

- Similar counterexamples can be found which are invariant under scalar multiplication.
- For Banach spaces with a basis, there is no pigeonhole principle even “up to $\epsilon$” for block sequences, with the (essentially) unique exception of $c_0$ (Odell & Schlumprecht, Gowers).
Games with block vectors

**Definition**

For $Y \in \text{bb}^\infty(E)$,

- $G[Y]$ denotes the **Gowers game** below $Y$: Players I and II alternate with I going first.
  - I plays $Y_n \preceq Y$,
  - II responds with vectors $y_n \in \langle Y_n \rangle$ such that $y_n < y_{n+1}$.

- $F[Y]$ denotes the **infinite asymptotic game** (due to Rosendal) below $Y$: Players I and II alternate with I going first
  - I plays $n_k \in \omega$,
  - II responds with vectors $y_k \in \langle Y \rangle$ such that $n_k < y_k < y_{k+1}$.

In both games, the **outcome** is the block sequence $(y_n)$.

- Plays of $F[Y]$ can be considered as plays of $G[Y]$ wherein I is restricted to playing “tail” block subsequences of $Y$. 

Theorem (Rosendal)

Whenever $\mathbb{A} \subseteq \mathbb{b}b^\infty(E)$ is analytic and $X \in \mathbb{b}b^\infty(E)$, there is a $Y \preceq X$ such that either

- I has a strategy in $F[Y]$ for playing into $\mathbb{A}^c$, or
- II has a strategy in $G[Y]$ for playing into $\mathbb{A}$.

If $Y$ is as in the theorem, then the first bullet implies that $\mathbb{A}^c$ is $\preceq$-dense below $Y$, while the second implies that $\mathbb{A}$ is.

This is a discrete and exact form of Gowers’ dichotomy for block sequences in Banach spaces, and implies it.
Local form?

Theorem (Rosendal)

Whenever $\mathbb{A} \subseteq \mathbb{bb}^\infty(E)$ is analytic and $X \in \mathbb{bb}^\infty(E)$, there is a $Y \preceq X$ such that either

- I has a strategy in $F[Y]$ for playing into $\mathbb{A}^c$, or
- II has a strategy in $G[Y]$ for playing into $\mathbb{A}$.

Our motivating question: Is there a local form?

Possible obstacles:

- What is a “coideal” of block sequences?
- Coideals on $\omega$ witness the pigeonhole principle. There is no pigeonhole principle here...
Families of block sequences

Definition

- **By a family** $\mathcal{H} \subseteq \text{bb}^\infty(E)$, we mean a non-empty set which is upwards closed with respect to $\preceq^*$. 

- A family $\mathcal{H} \subseteq \text{bb}^\infty(E)$ has the **(p)**-property if whenever $X_0 \succeq X_1 \succeq \cdots$ in $\mathcal{H}$, there is an $X \in \mathcal{H}$ such that $X \preceq^* X_n$ for all $n$. 

- A family $\mathcal{H} \subseteq \text{bb}^\infty(E)$ is **full** if whenever $D \subseteq E$ and $X \in \mathcal{H}$ is such that for all $Y \in \mathcal{H} \upharpoonright X$, there is $Z \preceq Y$ with $\langle Z \rangle \subseteq D$, then there is $Z \in \mathcal{H} \upharpoonright X$ with $\langle Z \rangle \subseteq D$. 

A full family with the (p)-property is a **(p^+)**-family.

- Fullness says that $\mathcal{H}$ witnesses the pigeonhole principle wherever it holds “$\mathcal{H}$-frequently” below an element of $\mathcal{H}$. 

- **(p^+)**-filters can be obtained by forcing with $(\text{bb}^\infty(E), \preceq^*)$, or built under CH or MA. Their existence is independent of ZFC.
A local Rosendal dichotomy

Theorem (S.)

Let $\mathcal{H} \subseteq \mathbb{b}b^\infty(E)$ be a $(p^+)$-family. Then, whenever $\mathbb{A} \subseteq \mathbb{b}b^\infty(E)$ is analytic and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \restriction X$ such that either

- I has a strategy for playing $F[Y]$ into $\mathbb{A}^c$, or
- II has a strategy for playing $G[Y]$ into $\mathbb{A}$.

The proof closely follows Rosendal’s, using “combinatorial forcing” to obtain the result for open sets.

Fullness is necessary; it is implied by the theorem for open sets.

A caveat: the second conclusion of the theorem does not appear sufficient to determine whether $\mathcal{H} \restriction X$ meets $\mathbb{A}$. 
The last concern is addressed with the following:

**Definition**

A family $\mathcal{H} \subseteq \mathbb{b}b^\infty(E)$ is **strategic** if whenever $X \in \mathcal{H}$ and $\alpha$ is a strategy for II in $G[X]$, then there is an outcome of $\alpha$ in $\mathcal{H}$.

- Strategies for II are (a priori) complicated objects, however the set of outcomes can be refined to a $\preceq$-dense closed set.
- Strategic $(p^+)$-filters can be obtained similarly as $(p^+)$-filters.
Extending to $\mathbf{L}(\mathbb{R})$

**Theorem (S.)**

Assume $\exists$ supercompact $\kappa$. Let $\mathcal{H} \subseteq \mathbb{bb}^\infty(E)$ be a strategic $(p^+)$-family. Then, whenever $\mathbb{A} \subseteq \mathbb{bb}^\infty(E)$ is in $\mathbf{L}(\mathbb{R})$ and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either

- I has a strategy for playing $F[Y]$ into $\mathbb{A}^c$, or
- II has a strategy for playing $G[Y]$ into $\mathbb{A}$.

**Corollary (S.)**

Assume $\exists$ supercompact $\kappa$. A filter $\mathcal{F} \subseteq \mathbb{bb}^\infty(E)$ is $\mathbf{L}(\mathbb{R})$-generic for $(\mathbb{bb}^\infty(E), \preceq^*)$ if and only if it is a strategic $(p^+)$-filter.

- The theorem is proved first for filters, and generalized by forcing with a given strategic $(p^+)$-family to add a strategic $(p^+)$-filter without adding reals.
Our proof uses the following Mathias-like notion of forcing:

**Definition**

\( \mathbb{P} \) is the set of all triples \((\vec{x}, X, \sigma)\) where \(\vec{x}\) is a finite block sequence, \(\vec{x} < X \in \text{bb}\^{\infty}(E)\), and \(\sigma\) is a strategy for \(I\) in \(F[X]\). \((\vec{y}, Y, \tau) \leq (\vec{x}, X, \sigma)\) if:

- \(\vec{y}\) is an extension of \(\vec{x}\) by plays of \(II\) against \(\sigma\) in \(F[X]\),
- \(Y \preceq X\), and
- On their shared domain, \(\tau\) is pointwise \(\succeq\) than \(\sigma\).

For \(\mathcal{H} \subseteq \text{bb}\^{\infty}(E)\), we write \(\mathbb{P}(\mathcal{H})\) for the set of all \((\vec{x}, X, \sigma)\) with \(X \in \mathcal{H}\).

- When \(\mathcal{U}\) is a \((p^+)\)-filter, \(\mathbb{P}(\mathcal{U})\) is c.c.c., and satisfies a very weak form of the “pure extension property”.
- This is used in conjunction with the fact that sets in \(\mathbb{L}(\mathbb{R})\) are universally Baire under our large cardinal hypothesis.
Thanks for listening!

Happy Families.

Large cardinals imply that every reasonably definable set of reals is Lebesgue measurable.

Semiselective coideals.

An infinite Ramsey theorem and some Banach-space dichotomies.

[5] Rosendal, C.
An exact Ramsey principle for block sequences.

[6] Smythe, I. B.
A local Ramsey theory for block sequences.
*In preparation.*