Towards a selective (local) Gowers dichotomy

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Outline

1. Review of (local) Ramsey theory on ω
2. Ramsey theory for block sequences in vector spaces
3. Local Ramsey theory for block sequences in vector spaces

Ramsey's theorem

Theorem (Ramsey)

For every $n \in \omega$*, if* $A \subseteq [\omega]^n$ *and* $X \in [\omega]^\omega$*, then there is a* $Y \in [X]^\omega$ *such that either* $[Y]^n \cap A = \emptyset$ *or* $[Y]^n \subseteq A$*.*

- $[X]^n$ is the set of all n-element subsets of X, for $n \in \omega \cup \{\omega\}$.
- Good exercise in recursive constructions of length $2^{\aleph_0}$:
  The theorem is false if $n = \omega$.

Infinite dimensional Ramsey theory

By putting definability restrictions on the partition, we obtain:

Theorem (Silver)

If $\mathbb{A} \subseteq [\omega]^\omega$ *is analytic and* $X \in [\omega]^\omega$*, then there is a* $Y \in [X]^\omega$ *such that either* $[Y]^\omega \cap \mathbb{A} = \emptyset$ *or* $[Y]^\omega \subseteq \mathbb{A}$*.*

With more assumptions, we can go well beyond the analytic sets:

Theorem (Shelah & Woodin)

Assume $\exists$ supercompact κ. *If* $\mathbb{A} \subseteq [\omega]^\omega$ *is in* $\mathbf{L}(\mathbb{R})$ *and* $X \in [\omega]^\omega$*, then there is a* $Y \in [X]^\omega$ *such that either* $[Y]^\omega \cap \mathbb{A} = \emptyset$ *or* $[Y]^\omega \subseteq \mathbb{A}$*.*

Local Ramsey theory

Theorem (Silver, Shelah & Woodin)

($\exists$ supercompact κ.) *If* $\mathbb{A} \subseteq [\omega]^\omega$ *is analytic (in* $\mathbf{L}(\mathbb{R})$*), then for any* $X \in [\omega]^\omega$*, there is a* $Y \in [X]^\omega$ *such that either* $[Y]^\omega \cap \mathbb{A} = \emptyset$ *or* $[Y]^\omega \subseteq \mathbb{A}$*.*

Local Ramsey theory concerns "localizing" the witness Y above. That is, finding families $\mathcal{H} \subseteq [\omega]^\omega$ such that, provided the given X is in $\mathcal{H}$, Y can also be found in $\mathcal{H}$.

Local Ramsey theory (cont'd)

Definition

- $\mathcal{H} \subseteq [\omega]^\omega$ is a coideal if it is the complement of a (non-trivial) ideal. Equivalently, it is a non-empty family such that
  - $X \in \mathcal{H}$ and $X \subseteq^* Y \implies Y \in \mathcal{H}$,
  - $X, Y \in [\omega]^\omega$ with $X \cup Y \in \mathcal{H} \implies X \in \mathcal{H}$ or $Y \in \mathcal{H}$.
- A coideal $\mathcal{H} \subseteq [\omega]^\omega$ is selective if whenever $X_0 \supseteq X_1 \supseteq \cdots$ are in $\mathcal{H}$, there is an $X \in \mathcal{H}$ such that $X/n \subseteq X_n$ for all $n \in X$.

Examples (of selective coideals)

- $[\omega]^\omega$
- $\mathcal{U}$ a selective (or sufficiently generic) ultrafilter
- $[\omega]^\omega \setminus \mathcal{I}$ where $\mathcal{I}$ is the ideal generated by an infinite a.d. family

Local Ramsey theory (cont'd)

Theorem (Mathias, Todorcevic)

($\exists$ supercompact κ.) *Let $\mathcal{H} \subseteq [\omega]^\omega$ be a selective coideal. If $\mathbb{A} \subseteq [\omega]^\omega$ is analytic (in $\mathbf{L}(\mathbb{R})$), then for any $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either $[Y]^\omega \cap \mathbb{A} = \emptyset$ or $[Y]^\omega \subseteq \mathbb{A}$.*

Corollary

- ($\exists$ supercompact κ.) *If $\mathcal{A}$ is an infinite a.d. family which is analytic (in $\mathbf{L}(\mathbb{R})$), then $\mathcal{A}$ fails to be maximal.*
- ($\exists$ supercompact κ.) *A filter $\mathcal{G}$ is $\mathbf{L}(\mathbb{R})$-generic for $([\omega]^\omega, \subseteq^*)$ if and only if $\mathcal{G}$ is selective.*

- As a result, selective ultrafilters are said to have "complete combinatorics" (see work of Blass, LaFlamme, Dobrinen)
- An "abstract" version has recently been developed for topological Ramsey spaces (by Di Prisco, Mijares, & Nieto).

Block sequences in vector spaces

Let E be an $\aleph_0$-dimensional vector space over $\mathbb{Q}$, with basis (e_n).

Definition

- Given any vector $x = a_0 e_0 + \cdots + a_k e_k$, its support (with respect to (e_n)) is $\mathrm{supp}(x) = \{n : a_n \neq 0\}$.
- A block sequence (with respect to (e_n)) is a sequence (x_n) of vectors such that $\max(\mathrm{supp}(x_n)) < \min(\mathrm{supp}(x_{n+1}))$, written $x_n < x_{n+1}$, for all n.
- For X and Y block sequences, if X is block with respect to Y, we write $X \preceq Y$. Equivalently (for *block* sequences), $\langle X \rangle \subseteq \langle Y \rangle$.

Let $\mathrm{bb}^\infty(E)$ be the (Polish) space of infinite block sequences in E.

- Abuse of terminology: "vectors" = non-zero vectors.
- $(\mathrm{bb}^\infty(E), \preceq^*)$ (i.e., $\preceq$ modulo finite) is a σ-closed poset, equivalent to forcing with infinite dimensional subspaces of E.

Ramsey theory for block sequences?

What would a Ramsey theorem block sequences in E look like?

A "pigeonhole principle": If $A \subseteq E$, there is an $X \in \mathrm{bb}^{\infty}(E)$ all of whose ∞-dimensional (block) subspaces are contained in one of A or A^c.

Example

This is false. Let A be vectors whose first coefficient, with respect to the basis (e_n), is positive. There is no X with the above property.

- Similar counterexamples can be found which are invariant under scalar multiplication.
- For Banach spaces with a basis, there is no pigeonhole principle even "up to ϵ" for block sequences, with the (essentially) unique exception of c_0 (Odell & Schlumprect, Gowers).

Games with block vectors

Definition

For $Y \in \mathrm{bb}^\infty(E)$,

- $G[Y]$ denotes the Gowers game below Y: Players I and II alternate with I going first.
  - I plays $Y_n \preceq Y$,
  - II responds with vectors $y_n \in \langle Y_n \rangle$ such that $y_n < y_{n+1}$.
- $F[Y]$ denotes the infinite asymptotic game (due to Rosendal) below Y: Players I and II alternate with I going first
  - I plays $n_k \in \omega$,
  - II responds with vectors $y_k \in \langle Y \rangle$ such that $n_k < y_k < y_{k+1}$.

In both games, the outcome is the block sequence (y_n).

- Plays of $F[Y]$ can be considered as plays of $G[Y]$ wherein I is restricted to playing "tail" block subsequences of Y.

Rosendal's dichotomy

Theorem (Rosendal)

Whenever $\mathbb{A} \subseteq \mathrm{bb}^{\infty}(E)$ is analytic and $X \in \mathrm{bb}^{\infty}(E)$, there is a $Y \preceq X$ such that either

- *I has a strategy in $F[Y]$ for playing into $\mathbb{A}^c$, or*
- *II has a strategy in $G[Y]$ for playing into $\mathbb{A}$.*

- If Y is as in the theorem, then the first bullet implies that $\mathbb{A}^c$ is $\preceq$-dense below Y, while the second implies that $\mathbb{A}$ is.
- This is a discrete and exact form of Gowers' dichotomy for block sequences in Banach spaces, and implies it.

Local form?

Theorem (Rosendal)

Whenever $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ is analytic and $X \in \mathrm{bb}^\infty(E)$, there is a $Y \preceq X$ such that either

- *I has a strategy in $F[Y]$ for playing into $\mathbb{A}^c$, or*
- *II has a strategy in $G[Y]$ for playing into $\mathbb{A}$.*

Our motivating question: Is there a local form?

Possible obstacles:

- What is a “coideal” of block sequences?
- Coideals on ω witness the pigeonhole principle. There is no pigeonhole principle here...

Families of block sequences

Definition

- By a family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$, we mean a non-empty set which is upwards closed with respect to $\preceq^*$.
- A family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ has the (p)-property if whenever $X_0 \succeq X_1 \succeq \cdots$ in $\mathcal{H}$, there is an $X \in \mathcal{H}$ such that $X \preceq^* X_n$ for all n.
- A family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ is full if whenever $D \subseteq E$ and $X \in \mathcal{H}$ is such that for all $Y \in \mathcal{H} \restriction X$, there is $Z \preceq Y$ with $\langle Z \rangle \subseteq D$, then there is $Z \in \mathcal{H} \restriction X$ with $\langle Z \rangle \subseteq D$.

A full family with the (p)-property is a (p^+)-family.

- Fullness says that $\mathcal{H}$ witnesses the pigeonhole principle wherever it holds "$\mathcal{H}$-frequently" below an element of $\mathcal{H}$.
- (p^+)-filters can be obtained by forcing with $(\mathrm{bb}^\infty(E), \preceq^*)$, or built under CH or MA. Their existence is independent of ZFC.

A local Rosendal dichotomy

Theorem (S.)

Let $\mathcal{H} \subseteq \mathrm{bb}^{\infty}(E)$ be a (p^+)-family. Then, whenever $\mathbb{A} \subseteq \mathrm{bb}^{\infty}(E)$ is analytic and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either

- *I has a strategy for playing $F[Y]$ into $\mathbb{A}^c$, or*
- *II has a strategy for playing $G[Y]$ into $\mathbb{A}$.*

- The proof closely follows Rosendal's, using "combinatorial forcing" to obtain the result for open sets.
- Fullness is necessary; it is implied by the theorem for open sets.
- A caveat: the second conclusion of the theorem does not appear sufficient to determine whether $\mathcal{H} \upharpoonright X$ meets $\mathbb{A}$.

A local Rosendal dichotomy (cont'd)

The last concern is addressed with the following:

Definition

A family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ is strategic if whenever $X \in \mathcal{H}$ and α is a strategy for II in $G[X]$, then there is an outcome of α in $\mathcal{H}$.

- Strategies for II are (a priori) complicated objects, however the set of outcomes can be refined to a $\preceq$-dense closed set.
- Strategic (p^+)-filters can be obtained similarly as (p^+)-filters.

Extending to $\mathbf{L}(\mathbb{R})$

Theorem (S.)

Assume $\exists$ supercompact κ. *Let* $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ *be a strategic* (p^+)*-family. Then, whenever* $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ *is in* $\mathbf{L}(\mathbb{R})$ *and* $X \in \mathcal{H}$*, there is a* $Y \in \mathcal{H} \upharpoonright X$ *such that either*

- *I has a strategy for playing* $F[Y]$ *into* $\mathbb{A}^c$*, or*
- *II has a strategy for playing* $G[Y]$ *into* $\mathbb{A}$*.*

Corollary (S.)

Assume $\exists$ supercompact κ. *A filter* $\mathcal{F} \subseteq \mathrm{bb}^\infty(E)$ *is* $\mathbf{L}(\mathbb{R})$*-generic for* $(\mathrm{bb}^\infty(E), \preceq^*)$ *if and only if it is a strategic* (p^+)*-filter.*

- The theorem is proved first for filters, and generalized by forcing with a given strategic (p^+)-family to add a strategic (p^+)-filter without adding reals.

Extending to $\mathbf{L}(\mathbb{R})$ (cont'd)

Our proof uses the following Mathias-like notion of forcing:

Definition

$\mathbb{P}$ is the set of all triples $(\vec{x}, X, \sigma)$ where $\vec{x}$ is a finite block sequence, $\vec{x} < X \in \mathrm{bb}^\infty(E)$, and σ is a strategy for I in $F[X]$. $(\vec{y}, Y, \tau) \leq (\vec{x}, X, \sigma)$ if

- $\vec{y}$ is an extension of $\vec{x}$ by plays of II against σ in $F[X]$,
- $Y \preceq X$, and
- On their shared domain, τ is pointwise $\geq$ than σ.

For $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$, we write $\mathbb{P}(\mathcal{H})$ for the set of all $(\vec{x}, X, \sigma)$ with $X \in \mathcal{H}$.

- When $\mathcal{U}$ is a (p^+)-filter, $\mathbb{P}(\mathcal{U})$ is c.c.c., and satisfies a *very* weak form of the "pure extension property".
- This is used in conjunction with the fact that sets in $\mathbf{L}(\mathbb{R})$ are universally Baire under our large cardinal hypothesis.

Thanks for listening!

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In preparation.