A local Ramsey theory for block sequences

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Outline

1. Review of (local) Ramsey theory on $\mathbb{N}$
2. Ramsey theory for block sequences in vector spaces
3. Local Ramsey theory for block sequences in vector spaces
4. Projections in the Calkin algebra

Infinite dimensional Ramsey theory

Theorem (Silver, 1970)

If $\mathbb{A} \subseteq [\mathbb{N}]^\infty$ *is analytic and* $X \in [\mathbb{N}]^\infty$*, then there is a* $Y \in [X]^\infty$ *such that either* $[Y]^\infty \cap \mathbb{A} = \emptyset$ *or* $[Y]^\infty \subseteq \mathbb{A}$*.*

- Here, $[X]^\infty$ is the set of all infinite subsets of X.
- This result was the culmination of work of Ramsey, Nash-Williams, Galvin, and Prikry.

Infinite dimensional Ramsey theory

With more assumptions, we can go well beyond the analytic sets:

Theorem (Shelah & Woodin, 1990)

Assume $\exists$ supercompact κ. *If* $\mathbb{A} \subseteq [\mathbb{N}]^\infty$ *is in* $\mathbf{L}(\mathbb{R})$ *and* $X \in [\mathbb{N}]^\infty$, *then there is a* $Y \in [X]^\infty$ *such that either* $[Y]^\infty \cap \mathbb{A} = \emptyset$ *or* $[Y]^\infty \subseteq \mathbb{A}$.

Local Ramsey theory

Theorem (Silver, 1970 (Shelah & Woodin, 1990))

(Assume $\exists$ supercompact κ.) *If $\mathbb{A} \subseteq [\mathbb{N}]^\infty$ is analytic (in $\mathbf{L}(\mathbb{R})$) and $X \in [\mathbb{N}]^\infty$, then there is a $Y \in [X]^\infty$ such that either $[Y]^\infty \cap \mathbb{A} = \emptyset$ or $[Y]^\infty \subseteq \mathbb{A}$.*

Local Ramsey theory concerns "localizing" the witness Y above. That is, finding families $\mathcal{H} \subseteq [\mathbb{N}]^\infty$ such that, provided the given X is in $\mathcal{H}$, Y can also be found in $\mathcal{H}$.

Local Ramsey theory (cont'd)

Definition

- $\mathcal{H} \subseteq [\mathbb{N}]^\infty$ is a coideal if it is the complement of a (non-trivial) ideal. Equivalently, it is a non-empty family such that
  - $X \in \mathcal{H}$ and $X \subseteq^* Y \implies Y \in \mathcal{H}$,
  - $X, Y \in [\mathbb{N}]^\infty$ with $X \cup Y \in \mathcal{H} \implies X \in \mathcal{H}$ or $Y \in \mathcal{H}$.
- A coideal $\mathcal{H} \subseteq [\mathbb{N}]^\infty$ is selective (or a happy family) if whenever $X_0 \supseteq X_1 \supseteq \cdots$ are in $\mathcal{H}$, there is an $X \in \mathcal{H}$ such that $X/n \subseteq X_n$ for all $n \in X$.

Examples (of selective coideals)

- $[\mathbb{N}]^\infty$
- $\mathcal{U}$ a selective (or sufficiently generic) ultrafilter
- $[\mathbb{N}]^\infty \setminus \mathcal{I}$ where $\mathcal{I}$ is the ideal generated by an infinite a.d. family

Local Ramsey theory (cont'd)

Theorem (Mathias, 1977 (Todorcevic, 1997))

(Assume $\exists$ supercompact κ.) *Let $\mathcal{H} \subseteq [\mathbb{N}]^\infty$ be a selective coideal. If $\mathbb{A} \subseteq [\mathbb{N}]^\infty$ is analytic (in $\mathbf{L}(\mathbb{R})$), then for any $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either $[Y]^\infty \cap \mathbb{A} = \emptyset$ or $[Y]^\infty \subseteq \mathbb{A}$.*

Corollary

Assume $\exists$ supercompact κ. *A filter $\mathcal{G}$ is $\mathbf{L}(\mathbb{R})$-generic for $([\mathbb{N}]^\infty, \subseteq^*)$ if and only if $\mathcal{G}$ is selective.*

- Selective ultrafilters are said to have "complete combinatorics" (cf. work of Blass, LaFlamme, Dobrinen)
- An "abstract" version has recently been developed for topological Ramsey spaces (Di Prisco, Mijares, & Nieto, 2015).

Block sequences in vector spaces

Let B be a Banach space with normalized Schauder basis (e_n), and $E = \mathrm{span}_F(e_n)$, for F a countable subfield of $\mathbb{R}$ (or $\mathbb{C}$) so that the norm on E takes values in F.

Definition

- Given any vector x in B, its support (with respect to (e_n)) is $\mathrm{supp}(x) = \{k : x = \sum_n a_n e_n \Rightarrow a_k \neq 0\}$. Write $x < y$ if $\max(\mathrm{supp}(x)) < \min(\mathrm{supp}(y))$.
- If $\mathrm{supp}(x)$ is finite, then x is a block vector.
- A block sequence (with respect to (e_n)) is a sequence of vectors (x_n) such that $x_0 < x_1 < x_2 < \cdots$.
- For X and Y block sequences, if X is block with respect to Y, write $X \preceq Y$. Equivalently (for *block* sequences), $\mathrm{span}(X) \subseteq \mathrm{span}(Y)$.
- Let $\mathrm{bb}^\infty(B)$ be the space of infinite normalized block sequences in B, a Polish subspace of $B^{\mathbb{N}}$. Similarly for $\mathrm{bb}^\infty(E)$.

Ramsey theory for block sequences?

What would a Ramsey theorem block sequences in E look like?

A "pigeonhole principle": If $A \subseteq E$, there is an $X \in \mathrm{bb}^\infty(E)$ all of whose ∞-dimensional (block) subspaces are contained in one of A or A^c.

Example

This is false. Let A be vectors whose first coefficient, with respect to the basis (e_n), is positive. There is no X with the above property.

- Similar counterexamples can be found which are invariant under scalar multiplication.
- For general Banach spaces B, there is no pigeonhole principle even "up to ϵ" for block sequences, with the (essentially) unique exception of c_0 (Gowers, 1992).

Games with block vectors

Definition

For $Y \in \mathrm{bb}^\infty(E)$,

- $G[Y]$ denotes the Gowers game below Y: Players I and II alternate with I going first.
  - I plays $Y_k \preceq Y$,
  - II responds with a vector $y_k \in \mathrm{span}_F(Y_k)$ such that $y_k < y_{k+1}$.
- $F[Y]$ denotes the infinite asymptotic game below Y: Players I and II alternate with I going first
  - I plays $n_k \in \mathbb{N}$,
  - II responds with a vector $y_k \in \mathrm{span}_F(Y)$ such that $n_k < y_k < y_{k+1}$.

In both games, the outcome is the block sequence (y_k).

- For $Y \in \mathrm{bb}^\infty(B)$, the games are defined similarly, with II playing block vectors. We denote these games $G^*[Y]$ and $F^*[Y]$.

Gowers' dichotomy

Theorem (Gowers, 1996)

Whenever $\mathbb{A} \subseteq \mathrm{bb}^\infty(B)$ *is analytic,* $X \in \mathrm{bb}^\infty(B)$*, and* $\Delta = (\delta_n) > 0$*, then there is a* $Y \preceq X$ *such that either*

- *every* $Z \preceq Y$ *is in* $\mathbb{A}^c$*, or*
- *II has a strategy in* $G^*[Y]$ *for playing into* $\mathbb{A}_\Delta$*.*

- $\mathbb{A}_\Delta = \{(z_n) \in \mathrm{bb}^\infty(B) : \exists (z_n') \in \mathbb{A} \forall n (\|z_n - z_n'\| < \delta_n)\}$ is the Δ-expansion of $\mathbb{A}$.
- Assuming $\exists$ supercompact κ, this can be extended to sets $\mathbb{A}$ in $\mathbf{L}(\mathbb{R})$ (Lopez-Abad, 2005).

Rosendal's dichotomy

In the discrete setting, we have the following exact result:

Theorem (Rosendal, 2010)

Whenever $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ is analytic and $X \in \mathrm{bb}^\infty(E)$, there is a $Y \preceq X$ such that either

- *I has a strategy in $F[Y]$ for playing into $\mathbb{A}^c$, or*
- *II has a strategy in $G[Y]$ for playing into $\mathbb{A}$.*

- This can be used to prove Gowers' dichotomy, with minimal use of Δ-expansions.

Local forms?

Motivating question: Are there local forms of Gowers' and Rosendal's dichotomies?

Possible obstacles:

- What is a "coideal" of block sequences?
- Coideals on $\mathbb{N}$ witness the pigeonhole principle. There is no pigeonhole principle here...

Families of block sequences

Definition

- By a family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$, we mean a non-empty set which is upwards closed with respect to $\preceq^*$.
- A family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ has the (p)-property if whenever $X_0 \succeq X_1 \succeq \cdots$ in $\mathcal{H}$, there is an $X \in \mathcal{H}$ such that $X \preceq^* X_n$ for all n.
- A family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ is full if whenever $D \subseteq E$ and $X \in \mathcal{H}$ is such that for all $Y \in \mathcal{H} \restriction X$, there is $Z \preceq Y$ with $\langle Z \rangle \subseteq D$, then there is $Z \in \mathcal{H} \restriction X$ with $\langle Z \rangle \subseteq D$.

A full family with the (p)-property is a (p^+)-family.

- Fullness says that $\mathcal{H}$ witnesses the pigeonhole principle wherever it holds "$\mathcal{H}$-frequently" below an element of $\mathcal{H}$.
- (p^+)-filters can be obtained by forcing with $(\mathrm{bb}^\infty(E), \preceq^*)$, or built under CH or MA. Their existence is independent of ZFC.

A local Rosendal dichotomy

Theorem (S.)

Let $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ be a (p^+)-family. Then, whenever $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ is analytic and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either

- *I has a strategy for playing $F[Y]$ into $\mathbb{A}^c$, or*
- *II has a strategy for playing $G[Y]$ into $\mathbb{A}$.*

- The proof closely follows Rosendal's, using "combinatorial forcing" to obtain the result for open sets.
- Fullness is necessary; it is implied by the theorem for clopen sets.
- A caveat: the second conclusion of the theorem does not appear sufficient to determine whether $\mathcal{H} \upharpoonright X$ meets $\mathbb{A}$.

A local Rosendal dichotomy (cont'd)

The last concern is addressed with the following:

Definition

A family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ is strategic if whenever $X \in \mathcal{H}$ and α is a strategy for II in $G[X]$, then there is an outcome of α in $\mathcal{H}$.

- Strategies for II are (a priori) complicated objects, however the set of outcomes can be refined to a $\preceq$-dense closed set, using a lemma of Ferenczi & Rosendal.
- Strategic (p^+)-filters can be obtained similarly as (p^+)-filters.

Extending to $\mathbf{L}(\mathbb{R})$

Theorem (S.)

Assume $\exists$ supercompact κ. *Let $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ be a strategic (p^+)-family. Then, whenever $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ is in $\mathbf{L}(\mathbb{R})$ and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either*

- *I has a strategy for playing $F[Y]$ into $\mathbb{A}^c$, or*
- *II has a strategy for playing $G[Y]$ into $\mathbb{A}$.*

Corollary (S.)

Assume $\exists$ supercompact κ. *A filter $\mathcal{G} \subseteq \mathrm{bb}^\infty(E)$ is $\mathbf{L}(\mathbb{R})$-generic for $(\mathrm{bb}^\infty(E), \preceq^*)$ if and only if it is a strategic (p^+)-filter.*

- The theorem is proved first for filters, using a Mathias-like forcing, and generalized by forcing with a given strategic (p^+)-family to add a strategic (p^+)-filter without adding reals.

A local Gowers dichotomy

Theorem (S.)

(Assume $\exists$ supercompact κ.) *Let $\mathcal{H} \subseteq \mathrm{bb}^\infty(B)$ be a spread (strategic) (p^*)-family which is invariant under small perturbations. Then, whenever $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ is analytic (in $\mathbf{L}(\mathbb{R})$), $X \in \mathcal{H}$ and $\Delta > 0$, there is a $Y \in \mathcal{H} \restriction X$ such that either*

- *every $Z \preceq Y$ is in $\mathbb{A}^c$, or*
- *II has a strategy in $G^*[Y]$ for playing into $\mathbb{A}_\Delta$.*

- (p)-families in $\mathrm{bb}^\infty(B)$ are defined as before, and * denotes an approximate form of fullness.
- A family $\mathcal{H}$ is spread if each $X \in \mathcal{H}$ has a further $Y \in \mathcal{H} \restriction X$ whose supports are "spread out". Resembles a "(q)-property".
- A family is invariant under small perturbations if there is some $\Delta > 0$ so that $\mathcal{H}_\Delta = \mathcal{H}$.

Since the local Gowers dichotomy is approximate, the corresponding $\mathbf{L}(\mathbb{R})$-genericity result should be for a poset of block subspaces "modulo small perturbations". There are many options, we give one.

Projections in the Calkin algebra

Let H be a Hilbert space, with orthonormal basis (e_n).

The Calkin algebra is the quotient $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$, where $\mathcal{K}(H)$ is the ideal of compact operators.

Let $\mathcal{P}(\mathcal{C}(H))$ be the set of projections (those p with $p^2 = p^* = p$) in $\mathcal{C}(H)$.

$\mathcal{P}(\mathcal{C}(H))$ can be identified with the set of closed subspaces in H modulo compact perturbations, and inherits a natural ordering $\leq$.

Fact

- *If $\Delta > 0$ is summable, then a Δ-perturbation is a compact perturbation.*
- *The (images of) block projections are $\leq$-dense in $\mathcal{P}(\mathcal{C}(H))^+ = \mathcal{P}(\mathcal{C}(H)) \setminus \{0\}$.*

Projections in the Calkin algebra

Theorem (S.)

(Assume $\exists$ supercompact κ.) A filter $\mathcal{G} \subseteq \mathcal{P}(\mathcal{C}(H))^+$ is $\mathbf{L}(\mathbb{R})$-generic for $(\mathcal{P}(\mathcal{C}(H))^+, \leq)$ if and only if it is block dense and the corresponding set of block projections is a strategic (p^*)-family in $\mathrm{bb}^\infty(H)$.

- Why study such a notion of forcing?

Pure states on $\mathcal{B}(H)$

Definition

- A state on $\mathcal{B}(H)$ is a positive linear functional τ with $\tau(I) = 1$.
- A pure state is an extreme point in the (weak*-compact convex) set of states.

Example

If (e_n) is an orthonormal basis, and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$, then $\tau_{\mathcal{U}}(T) = \lim_{n\to\mathcal{U}}\langle Te_n, e_n\rangle$ defines a diagonalizable pure state.

- Anderson (1980) conjectured that every pure state on $\mathcal{B}(H)$ is diagonalizable.
- (Akemann & Weaver, 2008): (CH) There is a counterexample.
- (Farah & Weaver): Forcing with $(\mathcal{P}(\mathcal{C}(H))^+, \leq)$ produces a counterexample. (Uses the theory of quantum filters.)

Pure states on $\mathcal{B}(H)$ (cont'd)

While forcing over $\mathbf{L}(\mathbb{R})$ suffices to construct a non-diagonalizable pure state, and thus our characterization of $\mathbf{L}(\mathbb{R})$-generic filters applies, we can get away with less (and no large cardinals):

Theorem (S.)

If $\mathcal{F}$ is a quantum filter of projections in $\mathcal{P}(\mathcal{C}(H))^+$ which is block dense and the corresponding set of block projections is a spread (p^*)-family, then $\mathcal{F}$ yields a non-diagonalizable pure state.

- Such families $\mathcal{F}$ are easily constructed under CH or MA.
- One can show that any $\mathcal{F}$ satisfying the hypotheses of the theorem is a (genuine!) filter, but the existence of such families is independent of ZFC (Bice, 2011).
- The consistency of Anderson's conjecture remains unresolved.