Vectors and Dot Product

Name: Solutions

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira

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Review

(1) A *vector* $\mathbf{v}=\overrightarrow{PQ}$ is determined by a basepoint P and a terminal point Q.

(2) Components of $\mathbf{v}=\overrightarrow{PQ}$, where $P=(a_1,b_1)$ and $Q=(a_2,b_2)$:

$$\mathbf{v}=\langle a,b\rangle$$

with $a=a_2-a_1$ and $b=b_2-b_1$.

(3) The length $\|\mathbf{v}\|$ of $\mathbf{v}$ is equal to $\boxed{\sqrt{a^2+b^2}}^{(1)}$.

(4) *Vector addition:* $\langle v_1,v_2\rangle+\langle w_1,w_2\rangle=\boxed{\langle v_1+w_1,v_2+w_2\rangle}^{(2)}$.

(5) *Scalar multiplication:* $\|\lambda\mathbf{v}\|=|\lambda|\|\mathbf{v}\|$ for λ real.

(6) $\mathbf{v}$ and $\mathbf{w}$ are *parallel* if, for some scalar λ, $\boxed{\mathbf{w}=\lambda\mathbf{v}}^{(3)}$.

(7) If $\mathbf{v}$ makes an angle θ with the positive x-axis, then $v_1=\boxed{\|\mathbf{v}\|\cos\theta}^{(4)}$ and $v_2=\boxed{\|\mathbf{v}\|\sin\theta}^{(5)}$.

(8) *Equation of the sphere of radius R and center (a,b,c):* $\boxed{(x-a)^2+(y-b)^2+(z-c)^2=R^2}^{(6)}$.

(9) *Equation of the cylinder of radius R and vertical axis through (a,b,0):* $\boxed{(x-a)^2+(y-b)^2=R^2}^{(7)}$.

(10) *Equations for the line passing through $P_0=(x_0,y_0,z_0)$ with direction vector $\mathbf{v}=\langle a,b,c\rangle$:*

   (a) *Vector parametrization:* $\mathbf{r}(t)=\overrightarrow{OP_0}+t\mathbf{v}=\boxed{\langle x_0,y_0,z_0\rangle+t\langle a,b,c\rangle}^{(8)}$.

   (b) *Parametric equation:* $x=\boxed{x_0+at}^{(9)}$, $y=\boxed{y_0+bt}^{(10)}$, $z=\boxed{z_0+ct}^{(11)}$.

(11) The *dot product* of $\mathbf{v}=\langle a_1,b_1,c_1\rangle$ and $\mathbf{w}=\langle a_2,b_2,c_2\rangle$ is

$$\mathbf{v}\cdot\mathbf{w}=a_1a_2+b_1b_2+c_1c_2.$$

(12) $\mathbf{v}\cdot\mathbf{v}=\boxed{\|\mathbf{v}\|^2}^{(12)}$.

(13) Using the angle θ made by $\mathbf{u}$ and $\mathbf{v}$, we have $\mathbf{u}\cdot\mathbf{v}=\boxed{\|v\|\|w\|\cos\theta}^{(13)}$. The vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if $\boxed{\mathbf{u}\cdot\mathbf{v}=0}^{(14)}$.

(14) The angle between $\mathbf{v}$ and $\mathbf{w}$ is $\boxed{\text{acute}}^{(15)}$ if $\mathbf{v}\cdot\mathbf{w}>0$ and $\boxed{\text{obtuse}}^{(16)}$ if $\mathbf{v}\cdot\mathbf{w}<0$.

(15) Assume $\mathbf{v}\neq 0$. Every vector $\mathbf{u}$ has a decomposition $\mathbf{u}=\mathbf{u}_{||\mathbf{v}}+\mathbf{u}_{\perp\mathbf{v}}$, where $\mathbf{u}_{||\mathbf{v}}$ is parallel to $\mathbf{v}$, and $\mathbf{u}_{\perp\mathbf{v}}$ is perpendicular to $\mathbf{v}$. Explicitly,

$$\mathbf{u}_{||\mathbf{v}}=\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}=\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}=\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}\right)\mathbf{e}_{\mathbf{v}},\quad \mathbf{u}_{\perp\mathbf{v}}=\mathbf{u}-\mathbf{u}_{||\mathbf{v}},$$

where

$$\mathbf{e}_{\mathbf{v}}=\frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

(16) If $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}$, is it true that $\mathbf{v}=\mathbf{w}$? $\boxed{\text{No.}}^{(17)}$

Problems

(1) Let $R = (-2, 7)$. Calculate the following:

(a) The length of $\overrightarrow{OR}$.
Solution: $\sqrt{53}$.

(b) The components of $\mathbf{u} = \overrightarrow{PR}$, where $P = (1, 2)$.
Solution: $\langle -3, 5\rangle$.

(c) The point P such that $\overrightarrow{PR}$ has components $\langle -2, 7\rangle$.
Solution: $P = (0, 0)$.

(2) Find the given vector:

(a) Unit vector $\mathbf{e_v}$ where $\mathbf{v} = \langle 3, 4\rangle$.
Solution: $\langle \frac{3}{5}, \frac{4}{5}\rangle$.

(b) Vector of length 4 in the direction of of $\mathbf{u} = \langle -1, 1\rangle$.
Solution: $\langle -2\sqrt{2}, -2\sqrt{2}\rangle$

(c) Vector $\mathbf{v}$ of length 2 making an angle of 30° with the x-axis.
Solution: $\langle \sqrt{3}, 1\rangle$.

(3) Determine whether or not the two vectors are parallel:

(a) $\mathbf{u} = \langle 1, -2, 5\rangle$, $\mathbf{v} = \langle -2, 4, -10\rangle$.
Solution: Yes.

(b) $\mathbf{u} = \langle 4, 2, -6\rangle$, $\mathbf{v} = \langle 2, 1, 3\rangle$.
Solution: No.

(4) Find a vector parametrization for the line with the given description:

(a) Passes through $P = (1, 2, -8)$, direction vector $\mathbf{v} = \langle 2, 1, 3\rangle$.
Solution: $\mathbf{r}(t) = \langle 1 + 2t, 2 + t, -8 + 3t\rangle$.

(b) Passes through $(-2, 0, -2)$ and $(4, 3, 7)$.
Solution: $\mathbf{r}(t) = \langle -2 + 6t, 3t, -2 + 9t\rangle$.

(c) Passes through $(1, 1, 1)$ parallel to the line through $(2, 0, -1)$ and $(4, 1, 3)$.
Solution: $\mathbf{r}(t) = \langle 1 + 2t, 1 + t, 1 + 4t\rangle$.

(5) Show that the lines $\mathbf{r}_1(t) = \langle -1, 2, 2\rangle + t\langle 4, -2, 1\rangle$ and $\mathbf{r}_2(s) = \langle 0, 1, 1\rangle + s\langle 2, 0, 1\rangle$ do not intersect.
Solution: In class.

(6) Find the intersection of the lines $\mathbf{r}_1(t) = \langle -1, 1\rangle + t\langle 2, 4\rangle$ and $\mathbf{r}_2(s) = \langle 2, 1\rangle + s\langle -1, 6\rangle$ in the plane.
Solution: $(\frac{5}{4}, \frac{11}{2})$.

(7) Find all values of b for which the vectors are orthogonal.

(a) $\langle b, 3, 2\rangle, \quad \langle 1, b, 1\rangle$.
Solution: $b = -\frac{1}{2}$.

(b) $\langle 4, -2, 7\rangle, \quad \langle b^2, b, 0\rangle$.
Solution: $b = 0$ or $b = \frac{1}{2}$.

(8) Find the angle between $\mathbf{v}$ and $\mathbf{w}$ if $\mathbf{v} \cdot \mathbf{w} = \frac{1}{2}\|\mathbf{v}\|\|\mathbf{w}\|$.
Solution: $\frac{\pi}{3}$.

(9) If $\mathbf{e}$ and $\mathbf{f}$ are unit vectors and $\|\mathbf{e} + \mathbf{f}\| = \frac{3}{2}$, compute $\|\mathbf{e} - \mathbf{f}\|$.
Solution: $\frac{\sqrt{7}}{2}$.

(10) Find the projection of $\mathbf{u}$ along $\mathbf{v}$.

(a) $\mathbf{u} = \langle -1, 2, 0\rangle, \quad \mathbf{v} = \langle 2, 0, 1\rangle$.
Solution: $\langle -\frac{4}{5}, 0, -\frac{2}{5}\rangle$.

(b) $\mathbf{u} = 5\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}, \quad \mathbf{v} = \mathbf{k}$.
Solution: $-4\mathbf{k}$.

(11) Find the decomposition $\mathbf{a} = \mathbf{a}_{||\mathbf{b}} + \mathbf{a}_{\perp\mathbf{b}}$.

(a) $\mathbf{a} = \langle 4, -1, 5\rangle, \quad \mathbf{b} = \langle 2, 1, 1\rangle$.
SOLUTION: $\mathbf{a}_{||\mathbf{b}} = \langle 4, 2, 2\rangle$ and $\mathbf{a}_{\perp\mathbf{b}} = \langle 0, -3, 3\rangle$.

(b) $\mathbf{a} = \langle x, y\rangle, \quad \mathbf{b} = \langle 1, -1\rangle$.
SOLUTION: $\mathbf{a}_{||\mathbf{b}} = \langle \frac{x-y}{2}, \frac{y-x}{2}\rangle$ and $\mathbf{a}_{\perp\mathbf{b}} = \langle \frac{x+y}{2}, \frac{x+y}{2}\rangle$.