Differentiability and Tangent planes | Name: Solutions

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira | September 25, 2018

Review

(1) For small small changes Δx, Δy we have:

$$f(a+\Delta x, b) \approx f(a,b) + f_x(a,b)\Delta x$$

$$f(a, b+\Delta y) \approx f(a,b) + f_y(a,b)\Delta y$$

(2) Clairaut's theorem states that mixed partials are equal as long as all functions we are dealing with are continuous. Hence, we can take higher partial derivatives in any order we please.

(3) The linearization of$f$ in two and three variables:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(y-b)$$

$$L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

(4) If f_x and f_y exist and are continuous in a disk containing (a,b), then f is differetiable at (a,b).

(5) **Equation fo the tangent plane** to $z = f(x,y)$ at (a,b).

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(y-b)$$

(6) The **gradient** of a function f is fiven by $\nabla f = \left\langle \frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f, \frac{\partial}{\partial z} f \right\rangle$.

(7) **Chain rule for paths**: $\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f_{\mathbf{r}(t)} \cdot \mathbf{r}'(t)$

(8) The directional derivative with respect to $\mathbf{u}$ a unite vector is given by $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$

(9) If the angle between $\mathbf{u}$ and ∇f is θ, then $D_{\mathbf{u}} f = \|\nabla f\| \|\mathbf{u}\| \cos(\theta)$

(10) The equation of the tangent plane to the level surface $F(x,y,z) = k$ at point $P = (a,b,c)$ is

$$\nabla F_P \cdot \langle x-a, y-b, z-c \rangle = 0$$

Problems

1. The volume of a right-circular cone of radius r and height h is $V = \frac{\pi}{3}r^2h$. Use linear approximation to estimate the percentage change in volume of a right-circular cone of radius $r = 40$cm if the height is increased from 40 to 41cm.

   Solution: $\frac{\Delta V}{V} \approx \frac{(\partial V/\partial h)\cdot\Delta h}{V} = 0.025.$

2. Find f such that:

   (a) $\frac{\partial}{\partial x}f = 6x^2y$, $\quad \frac{\partial}{\partial y}f = 2x^3 - 3$

   (b) $\frac{\partial}{\partial x}f = e^x - y\sin(xy)$, $\quad \frac{\partial}{\partial y}f = -x\sin(xy) + 5y^4$

3. Find the points on the graph of $z = 3x^2 - 4y^2$ at which the vector $\mathbf{n} = \langle 3, 2, 2\rangle$ is normal to the tangent plane.

   Solution: $\left(-\frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$.

4. A bug located at $P = (3, 9, 4)$ begins walking in a straight line toward $(5, 7, 3)$. At what rate is the bug's temperature changing at P if the temperature is $T(x, y, z) = xe^{y-z}$? Units are in meters and degrees in Celcius.

   Solution: $-\frac{e^5}{3}$.

5. Determine the derivative of the function along the path given:

   (a) $f(x, y, z) = yx^2 - e^{xy} + ln(x)$ $\qquad \mathbf{r}(t) = \langle t^2, ln(t), \sqrt{t}\rangle$

   (b) $f(x, y, z) = \sin(xyz)$ $\qquad \mathbf{r}(t) = \langle e^t, \cos(4t), -t\rangle$

6. Find a function such that $\nabla f = \left\langle 2xe^y - z, y^3 + x^2e^y, -x\right\rangle$