CHAIN RULE AND OPTIMIZATION — NAME: SOLUTIONS

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira — October 2, 2018

# Review

(1) If $f$ is a function of $x, y, z$ and if $x, y, z$ depend on two other variables, say, $s$ and $t$, then

$$f(x, y, z) = f(x(s,t), y(s,t), z(s,t))$$

is the composite function of $s$ and $t$. We refer to $s$ and $t$ as the *independent variables*.

(2) The *Chain Rule* expresses the partial derivatives with respect to the independent variables:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s},$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}.$$

(3) The Chain Rule (for a function $f$ in $n$ variables) can be expressed as a dot product:

$$\frac{\partial f}{\partial t_k} = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle \cdot \left\langle \frac{\partial x_1}{\partial t_k}, \frac{\partial x_2}{\partial t_k}, \dots, \frac{\partial x_n}{\partial t_k} \right\rangle.$$

(4) *Implicit differentiation* is used to find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where $z$ is defined implicitly by an equation $F(x, y, z) = 0$:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z},$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

(5) We say that a point $P = (a, b)$ is a *critical point* of $f(x, y)$ if $f_x(a, b) = 0$ or does not exist, and $f_y(a, b) = 0$ or does not exist.

(6) The local minimum or maximum values of $f$ occur at critical points.

(7) The *discriminant* of $f(x, y)$ at $P = (a, b)$ is

$$D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}^2(a, b).$$

(8) *Second Derivative Test:* If $P = (a, b)$ is a critical point of $f(x, y)$, then

$$D(a,b) > 0, \quad f_{xx}(a,b) > 0 \Rightarrow \ f(a,b) \text{ is a local miminum}$$

$$D(a,b) > 0, \quad f_{xx}(a,b) < 0 \Rightarrow \ f(a,b) \text{ is a local maximum}$$

$$D(a,b) < 0 \Rightarrow \ \text{saddle point}$$

$$D(a,b) = 0 \Rightarrow \ \text{test inconclusive}$$

(9) If $f$ is continuous on a closed and bounded domain $\mathcal{D}$, then $f$ takes on both a minimum and maximum value on $\mathcal{D}$. The extreme values occur either at critical points in the interior of $\mathcal{D}$ or at points on the boundary of $\mathcal{D}$.

# Problems

(1) Use the Chain Rule to calculate the partial derivatives. Express the answer in terms of the independent variables.

(a) $\frac{\partial h}{\partial t_2}$; $h(x, y) = \frac{x}{y}$, $x = t_1 t_2$, $y = t_1^2 t_2$.
SOLUTION: $\frac{\partial h}{\partial t_2} = 0$.

(b) $\frac{\partial F}{\partial y}$; $F(u, v) = e^{u+v}$, $u = x^2$, $v = xy$.
SOLUTION: $\frac{\partial F}{\partial y} = xe^{x^2+xy}$.

(2) Suppose that $z$ is defined implicitly as a function of $x$ and $y$ by the equation $F(x, y, z) = xz^2 + y^2 z + xy - 1 = 0$.

(a) Calculate $F_x$, $F_y$, $F_z$.
SOLUTION: $F_x = z^2 + y$, $F_y = 2yz + x$, $F_z = 2xz + y^2$.

(b) Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
SOLUTION: $\frac{\partial z}{\partial x} = -\frac{z^2+y}{2xz+y^2}$, $\frac{\partial z}{\partial y} = -\frac{2yz+x}{2xz+y^2}$.

(3) Find the critical points of the following functions, then use the Second Derivative test to determine whether they are local minima, local maxima, or saddle points (or state that the test fails).

(a) $f(x, y) = 4x - 3x^3 - 2xy^2$.
SOLUTION: $(0, \pm\sqrt{2})$ are saddle points, $f(\frac{2}{3}, 0)$ is a local maximum, and $f(-\frac{2}{3}, 0)$ is a local minimum.

(b) $f(x, y) = \ln(x) + 2\ln(y) - x - 4y$.
SOLUTION: $f(1, \frac{1}{2})$ is a local maximum.

(4) Find the maximum of $f(x, y) = y^2 + xy - x^2$ on the square $0 \leq x \leq 2$, $0 \leq y \leq 2$.
SOLUTION: $f(1, 2) = 5$.