Vector Fields and Line Integrals

Name: Solutions

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira

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Important points

(1) A *vector field* assigns a vector to each point in a domain. A vector field in $\mathbb{R}^3$ is represented by

$$\mathbf{F} = \langle F_1, F_2, F_3 \rangle.$$

(2) The *divergence* of a vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is the scalar function given by

$$\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

(3) The *curl* of a vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is the vector field given by

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

(4) If $\mathbf{F} = \nabla f$, then f is called a *potential function* for $\mathbf{F}$. $\mathbf{F}$ is called *conservative* if it has a potential function.

(5) Any two potential functions for a conservative vector field differ by a constant (on an open, connected domain). In symbols, If $\mathbf{F} = \nabla f = \nabla g$, then $f - g = c$, where $c \in \mathbb{R}$.

(6) A conservative vector field $\mathbf{F}$ satisfies $\text{curl}(\mathbf{F}) = 0$.

(7) Line integral over a curve with parametrization $\mathbf{r}(t)$ for $a \leq t \leq b$:

- Scalar line integral:

$$\int_{\mathcal{C}} f(x, y, x) ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt.$$

- Vector line integral:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} (\mathbf{F} \cdot \mathbf{T}) ds = \int_a^b f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

Another notation: $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} F_1 dx + F_2 dy + F_3 dz$, for $F = \langle F_1, F_2, F_3 \rangle$.

(8) An *oriented curve* $\mathcal{C}$ is a curve in which one of the two possible directions along $\mathcal{C}$ (called the *positive direction*) is chosen.

(9) We write $-\mathcal{C}$ for the curve $\mathcal{C}$ with the opposite orientation. Then

$$\int_{-\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = -\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

(10) If $\delta(x, y, x)$ is the mass or charge density along $\mathcal{C}$, then the total mass or charge is equal to the scalar line integral $\int_{\mathcal{C}} \delta(x, y, z) ds$.

(11) The vector line integral is used to compute the work W exerted on an object along a curve $\mathcal{C}$:

$$W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

The work performed *against* $\mathbf{F}$ is the quantity $-\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

(12) Flux across $\mathcal{C} = \int_C (\mathbf{F} \cdot \mathbf{n}) ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{N}(t) dt$, where $\mathbf{N}(t) = \langle y'(t), -x'(t) \rangle$.

Problems

(1) Find a potential function for the vector field $\mathbf{F}$ by inspection or show that one does not exist.

(a) $\mathbf{F} = \langle 2xyz, x^2z, x^2yz\rangle$.
Solution: No potential function exists.

(b) $\mathbf{F} = \langle yz^2, xz^2, 2xyz\rangle$.
Solution: $\varphi(x, y, z) = xyz^2$.

(2) Prove the following identities:

(a) If f is a scalar function, then

$$\operatorname{div}(f\mathbf{F}) = f\operatorname{div}(\mathbf{F}) + \mathbf{F}\cdot\nabla f.$$

(b) $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}(\mathbf{F}) + (\nabla f)\times\mathbf{F}$.

(3) Evaluate the line integral.

(a) $\int_{\mathcal{C}} y dx - x dy$, parabola $y = x^2$ from $0 \le x \le 2$. Solution: -8/3.

(b) $\int_{\mathcal{C}} (x-y)dx + (y-z)dy + zdz$, line segment from $(0, 0, 0)$ to $(1, 4, 4)$. Solution: 13/2.

(4) Compute $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ for the oriented curve specified.

(a) $\mathbf{F}(x, y, z) = \left\langle \frac{1}{y^3+1}, \frac{1}{z+1}, 1\right\rangle$, $\mathbf{r}(t) = (t^3, 2, t^2)$ for $0 \le t \le 1$. Solution: 10/9.

(b) $\mathbf{F}(x, y, z) = \langle z^3, yz, x\rangle$, quarter of circle of radius 2 in the yz-plane with center at the origin where $y \ge 0$ and $z \ge 0$, oriented clockwise when viewed from the positive x-axis. Solution: 8/3.

(5) Find the total charge on a curve $y = x^{4/3}$ for $1 \le x \le 8$ (in centimeters) assuming a charge density of $\delta(x, y) = x/y$ (in units 10^{-6} C/cm). Solution: $\frac{1}{48}(73^{3/2} - 25^{3/2})$.

(6) Calculate the work done by a field $\mathbf{F} = \langle x + y, x - y\rangle$ when an object moves from $(0, 0)$ to $(1, 1)$ along each of the paths $y = x^2$ and $x = y^2$. Solution: 1 for both paths.

(7) Evaluate

$$\oint_{\mathcal{C}} \sin x\, dx + z\cos y\, dy + \sin y\, dz$$

where $\mathcal{C}$ is the ellipse $4x^2 + 9y^2 = 36$, oriented clockwise.

Solution: This vector field is conservative, with potential function $f(x, y, z) = z\sin y - \cos x$. This way, the integral is 0.