# More on Conservative Vector Fields

**Name: Solutions**

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira

November 8, 2018

## Review

(1) A vector field $\mathbf{F}$ on a domain $\mathcal{D}$ is called *path-independent* if for any two points $P, Q \in \mathcal{D}$, we have

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

for any two paths $\mathcal{C}_1$ and $C_2$ in $\mathcal{D}$ from $P$ to $Q$.

(2) The Fundamental Theorem for Conservative Vector Fields: If $\mathbf{F} = \nabla f$, then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f(Q) - f(P)$$

for any path $\mathbf{r}$ from $P$ to $Q$ in the domain of $\mathbf{F}$. This shows that conservative vector fields are path independent. The converse is also true: on an open, connected domain, a path-independent vector field is conservative.

(3) The work $W$ exerted on an object along a curve $\mathcal{C}$ is given by:

$$W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

The work performed *against* $\mathbf{F}$ is the quantity $-\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

# Problems

(1) Calculate the work required to move an object from $P = (1, 1, 1)$ to $Q = (3, -4, -2)$ against the force field $\mathbf{F}(x, y, z) = -12r^{-4}\langle x, y, z\rangle$, where $r = \sqrt{x^2 + y^2 + z^2}$.

Solution: $V = (x, y, z) = \frac{6}{x^2+y^2+z^2}$ is a potential function for $\mathbf{F}$. By the Fundamental Theorem,

$$W = -\int_{\overline{PQ}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_{\overline{PQ}} -\nabla V \cdot \mathrm{d}\mathbf{r} = -(V(Q) - V(P)) = \frac{52}{29}.$$

(2) Let $\mathbf{F}(x, y) = \langle 9y - y^3, e^{\sqrt{y}}(x^2 - 3x)\rangle$, and let $\mathcal{C}_2$ be the oriented curve in the picture below.

(a) Show that $\mathbf{F}$ is not conservative. Solution:

$$9 - 3y^2 = \frac{\partial \mathbf{F}_1}{\partial y} \neq \frac{\partial \mathbf{F}_2}{\partial x} = e^{\sqrt{y}}(2x - 3).$$

The cross partials are not equal, hence $\mathbf{F}$ is not conservative.

(b) Show that $\int_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ without explicitly computing this integral. *Hint:* Show that $\mathbf{F}$ is orthogonal to the edges along the square. Solution: On $\overline{OA}$, $y = 0$ hence $\mathbf{F} = \mathbf{F}(x, 0) = \langle 0, x^2 - 3x\rangle$, which is orthogonal to the vector in the direction of $\overline{OA}$, hence $\int_{\overline{OA}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$. Same idea for the other 3 segments.

(3) Find a conservative vector field of the form $\mathbf{F} = \langle g(y), h(x)\rangle$ such that $\mathbf{F}(0, 0) = \langle 1, 1\rangle$, where $g(y)$ and $h(x)$ are differentiable functions. Determine all such vector fields.

Solution: We need to find a scalar function $V(x, y)$ such that $\mathbf{F} = \nabla V$, that is:

$$\frac{\partial V}{\partial x} = g(y); \quad \frac{\partial V}{\partial y} = h(x)$$

Integrating the first one with respect to $x$, we get $V(x, y) = xg(y) + f(y)$. Differentiating this with respect to $y$ and comparing it to the second equation above,

$$\frac{\partial V}{\partial y} = xg'(y) + f'(y) = h(x)$$

This only holds when $g'(y)$ and $f'(y)$ are constants (take $x = 0$ to verify that). That it, $g'(y) = c_1$ and $f'(y) = c_2$, yielding $g(y) = c_1 y + d_1$ and $f(y) = c_2 y + d_2$. Substituting in the expression we found for $V$,

$$V(x, y) = x(c_1 y + d_1) + c_2 y + d_2 \Rightarrow \nabla V = \langle d_1 + c_1 y, c_2 + c_1 x\rangle.$$

By plugging in the condition $\mathbf{F}(0, 0) = \langle 1, 1\rangle$ we get $\mathbf{F}(x, y) = \langle 1 + by, 1 + bx\rangle$, for any $b \in \mathbb{R}$.

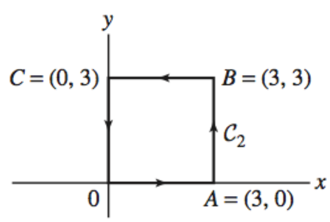


Figure 1: Problem 2