DIVERGENCE THEOREM REVIEW

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira

REVIEW

(1) Divergence of a vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$:

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

(2) **Divergence Theorem:** Let S be a closed surface that encloses a region W in \mathbb{R}^3 . Assume that S is piecewise smooth and is oriented by normal vectors pointing to the outside of W. Let \mathbf{F} be a vector field whose domain contains W. Then

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F}) \, dV.$$

- (3) If $\operatorname{div}(\mathbf{F}) = 0$, then \mathbf{F} has zero flux through the boundary $\partial \mathcal{W}$ of any \mathcal{W} contained in the domain of \mathbf{F} .
- (4) The divergence $\operatorname{div}(\mathbf{F})$ is interpreted as "flux per unit volume", which means that the flux through a small closed surface containing a point P is approximately equal to $\operatorname{div}(\mathbf{F})(P)$ times the enclosed volume.
- (5) Basic operations on functions and vector fields:

$$\begin{array}{cccc} f & \xrightarrow{\nabla} & \mathbf{F} & \xrightarrow{\operatorname{curl}} & \mathbf{G} & \xrightarrow{\operatorname{div}} g \\ \text{function} & & & \operatorname{vector field} & \xrightarrow{\operatorname{div}} g \end{array}$$

(6) The result of two consecutive operations is zero:

$$\operatorname{curl}(\nabla(f)) = \mathbf{0}, \qquad \operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0.$$

(7) The inverse-square field $\mathbf{F}_{\text{IS}} = \mathbf{e}_r/r^2$, defined for $r \neq 0$, satisfies div $(\mathbf{F}_{\text{IS}}) = 0$. The flux of \mathbf{F}_{IS} through a closed surface S is 4π if S contains the origin and is zero otherwise.

PROBLEMS

- (1) Which of the following is correct (\mathbf{F} is a continuously differentiable vector field defined everywhere)?
 - (a) The flux of $\operatorname{curl}(\mathbf{F})$ through all surfaces is (c) The flux of $\operatorname{curl}(\mathbf{F})$ through all closed surfaces zero. is zero
 - (b) If $\mathbf{F} = \nabla \phi$, then the flux of \mathbf{F} through all surfaces is zero. SOLUTION: (c).
- (2) How does the Divergence Theorem imply that the flux of $\mathbf{F} = \langle x^2, y e^z, y 2zx \rangle$ through a closed surface is equal to the enclosed volume? SOLUTION: Use the fact that $\operatorname{div}(\mathbf{F}) = 1$.
- (3) Apply the Divergence Theorem to evaluate the flux $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$.
 - (a) $\mathbf{F} = \langle xy^2, yz^2, zx^2 \rangle$, \mathcal{S} is the boundary of the cylinder given by $x^2 + y^2 \leq 4, \ 0 \leq z \leq 3$. Solution: 60π .
 - (b) $\mathbf{F} = \langle x + y, z, z x \rangle$, \mathcal{S} is the boundary of the
- (4) Calculate the flux of the vector field $\mathbf{F} = 2xy\mathbf{i} y^2\mathbf{j} + \mathbf{k}$ thorugh the surface S in figure 1. (*Hint:* Apply the Divergence Theorem to the closed surface consisting of \mathcal{S} and the unit disk). Solution: π .
- (5) Let $I = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x,y,z) = \left\langle \frac{2yz}{r^2}, -\frac{xz}{r^2}, -\frac{xy}{r^2} \right\rangle$$

 $(r = \sqrt{x^2 + y^2 + z^2})$ and \mathcal{S} is the boundary of a region \mathcal{W} .

- (a) Check that **F** is divergence-free. Direct computation involving SOLUTION: the single-variable chain rule.
- (b) Show that I = 0 if S is a sphere centered at the origin. Explain, however, why the Divergence Theorem cannot be used to prove this.

Solution: Use $I = \int_0^{2\pi} \int_0^{\pi} \mathbf{F}(\Phi(\theta, \phi)) \cdot \mathbf{N} \, d\phi d\theta$, where Φ is a parametrization for S in spherical coordinates. The Divergence Theorem cannot be used because \mathbf{F} is not defined at the origin, which is inside the ball with the boundary \mathcal{S} .



Figure 1: Problem 4.

region between the paraboloid $z = 9 - x^2 - y^2$ and the *xy*-plane.

Solution: 81π .