Divergence Theorem Review

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Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira

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Review

(1) Divergence of a vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$:

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

(2) **Divergence Theorem:** Let $\mathcal{S}$ be a closed surface that encloses a region $\mathcal{W}$ in $\mathbb{R}^3$. Assume that $\mathcal{S}$ is piecewise smooth and is oriented by normal vectors pointing to the outside of $\mathcal{W}$. Let $\mathbf{F}$ be a vector field whose domain contains $\mathcal{W}$. Then

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV.$$

(3) If $\operatorname{div}(\mathbf{F}) = 0$, then $\mathbf{F}$ has zero flux through the boundary $\partial\mathcal{W}$ of any $\mathcal{W}$ contained in the domain of $\mathbf{F}$.

(4) The divergence $\operatorname{div}(\mathbf{F})$ is interpreted as "flux per unit volume", which means that the flux through a small closed surface containing a point P is approximately equal to $\operatorname{div}(\mathbf{F})(P)$ times the enclosed volume.

(5) Basic operations on functions and vector fields:

$$\underset{\text{function}}{f} \xrightarrow{\nabla} \underset{\text{vector field}}{\mathbf{F}} \xrightarrow{\text{curl}} \underset{\text{vector field}}{\mathbf{G}} \xrightarrow{\text{div}} \underset{\text{function}}{g}$$

(6) The result of two consecutive operations is zero:

$$\operatorname{curl}(\nabla(f)) = \mathbf{0}, \qquad \operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0.$$

(7) The inverse-square field $\mathbf{F}_{\text{IS}} = \mathbf{e}_r/r^2$, defined for $r \neq 0$, satisfies $\operatorname{div}(\mathbf{F}_{\text{IS}}) = 0$. The flux of $\mathbf{F}_{\text{IS}}$ through a closed surface $\mathcal{S}$ is 4π if $\mathcal{S}$ contains the origin and is zero otherwise.

Problems

(1) Which of the following is correct ($\mathbf{F}$ is a continuously differentiable vector field defined everywhere)?

(a) The flux of curl($\mathbf{F}$) through all surfaces is zero.

(b) If $\mathbf{F} = \nabla\phi$, then the flux of $\mathbf{F}$ through all surfaces is zero.

(c) The flux of curl($\mathbf{F}$) through all closed surfaces is zero.

(2) How does the Divergence Theorem imply that the flux of $\mathbf{F} = \langle x^2, y - e^z, y - 2zx\rangle$ through a closed surface is equal to the enclosed volume?

(3) Apply the Divergence Theorem to evaluate the flux $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$.

(a) $\mathbf{F} = \langle xy^2, yz^2, zx^2\rangle$, $\mathcal{S}$ is the boundary of the cylinder given by $x^2 + y^2 \leq 4$, $0 \leq z \leq 3$.

(b) $\mathbf{F} = \langle x+y, z, z-x\rangle$, $\mathcal{S}$ is the boundary of the region between the paraboloid $z = 9-x^2-y^2$ and the xy-plane.

(4) Calculate the flux of the vector field $\mathbf{F} = 2xy\mathbf{i} - y^2\mathbf{j} + \mathbf{k}$ thorugh the surface $\mathcal{S}$ in figure 1. (*Hint:* Apply the Divergence Theorem to the closed surface consisting of $\mathcal{S}$ and the unit disk).

(5) Let $I = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = \left\langle \frac{2yz}{r^2}, -\frac{xz}{r^2}, -\frac{xy}{r^2} \right\rangle$$

($r = \sqrt{x^2 + y^2 + z^2}$) and $\mathcal{S}$ is the boundary of a region $\mathcal{W}$.

(a) Check that $\mathbf{F}$ is divergence-free.

(b) Show that $I = 0$ if $\mathcal{S}$ is a sphere centered at the origin. Explain, however, why the Divergence Theorem cannot be used to prove this.

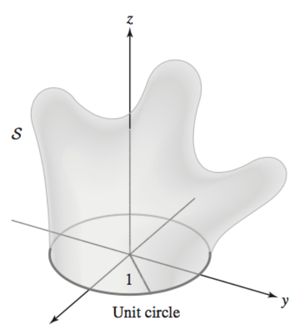

Figure 1: Problem 4.