Final Review

Name: Solutions

Math 1920 - Sections 221 and 222 - TA: Itamar Oliveira

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Review

(1) **Green's Theorem:** $\mathcal{D}$ is a domain in the plane and $\partial\mathcal{D}$ is its boundary. Then:

$$\oint_{\partial\mathcal{D}} F_1\,dx + F_2\,dy = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

or

$$\oint_{\partial\mathcal{D}} \mathbf{F}\cdot d\mathbf{r} = \iint_{\mathcal{D}} \operatorname{curl}_z(\mathbf{F})\,dA.$$

(2) **Stokes' Theorem:** $\mathcal{S}$ is an oriented surface and $\partial\mathcal{S}$ is its boundary. Then:

$$\oint_{\partial S} \mathbf{F}\cdot d\mathbf{r} = \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S}.$$

(3) **Divergence Theorem:** Let $\mathcal{S}$ be a closed surface that encloses a region $\mathcal{W}$ in $\mathbb{R}^3$. Assume that $\mathcal{S}$ is piecewise smooth and is oriented by normal vectors pointing to the outside of $\mathcal{W}$. Let $\mathbf{F}$ be a vector field whose domain contains $\mathcal{W}$. Then

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\,dV.$$

Problems

(1) Use Green's Theorem to evaluate the line integral around the given closed curve.

(a) $\oint_{\mathcal{C}} xy^3\,dx + x^3y\,dy$, where $\mathcal{C}$ is the rectangle $-1 \le x \le 2$, $-2 \le y \le 3$, oriented counterclockwise.
Solution: -30.

(b) $\oint_{\mathcal{C}} y^2\,dx - x^2\,dy$, where $\mathcal{C}$ consists of the arcs $y = x^2$ and $y = \sqrt{x}$, $0 \le x \le 1$, oriented clockwise.
Solution: $\frac{3}{5}$.

(2) Let I be the flux of $\mathbf{F} = \langle e^y, 2xe^{x^2}, z^2\rangle$ through the upper hemisphere $\mathcal{S}$ of the unit sphere.

(a) Let $\mathbf{G} = \langle e^y, 2xe^{x^2}, 0\rangle$. Find a vector field $\mathbf{A}$ such that $\mathrm{curl}(\mathbf{A}) = \mathbf{G}$.

Solution: $\mathbf{A} = \langle 0, 0, e^y - e^{x^2}\rangle$.

(b) Use Stokes' theorem to show that the flux of $\mathbf{G}$ through $\mathcal{S}$ is zero. *Hint:* Calculate the circulation of $\mathbf{A}$ around ∂S.
Solution: Use $\iint_S \mathbf{G}\cdot d\mathbf{S} = \oint_{\mathcal{C}} \mathbf{A}\cdot d\mathbf{r}$.

(c) Calculate I. *Hint:* Use (b) to show that I is equal to the flux of $\langle 0, 0, z^2\rangle$ through $\mathcal{S}$.
Solution: Use $\mathbf{F} = \mathrm{curl}(\mathbf{A}) + \langle 0, 0, z^2\rangle$.

(3) Let $\mathcal{S}$ be the portion of the plane $z = x$ contained in the half-cylinder of radius R. Use Stokes' theorem to calculate the circulation of $\mathbf{F} = \langle z, x, y + 2z\rangle$ around the boundary of $\mathcal{S}$ (a half-ellipse) in the counterclockwise direction when viewed from above.

Solution: Show that $\mathrm{curl}(\mathbf{F})$ is orthogonal to the normal vector to the plane. The circulation is zero.

(4) Show that the circulation of $\mathbf{F}(x, y, z) = \langle x^2, y^2, z(x^2 + y^2)\rangle$ around any curve $\mathcal{C}$ on the surface of the cone $z^2 = x^2 + y^2$ is equal to zero.

Solution: Show that $\mathrm{curl}(\mathbf{F})$ at a given point in the region enclosed by $\mathcal{C}$ is orthogonal to the normal vector and use Stokes.

(5) Compute the flux of $\mathbf{F} = \langle xyz + xy, \frac{1}{2}y^2(1 - z) + e^x, e^{x^2+y^2}\rangle$, $\mathcal{S}$ is the boundary of the solid bounded by the cylinder $x^2 + y^2 = 16$ and the planes $z = 0$ and $z = y - 4$.

Solution: -128π.

(6) Compute the flux of $\mathbf{F} = \langle \sin(yz), \sqrt{x^2 + z^4}, x\cos(x - y)\rangle$, $\mathcal{S}$ is any smooth closed surface that is the boundary of a region in $\mathbb{R}^3$.

Solution: 0.

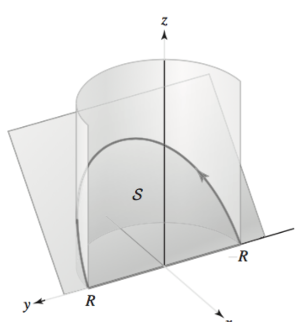

Figure 1: Problem 5.

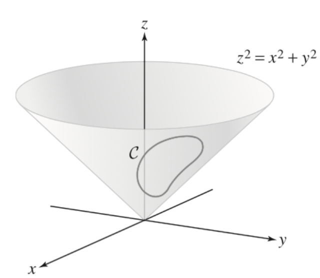

Figure 2: Problem 4.