Continuity (Part II) and more

Name: ____________

Math 1110 - Instructor: Itamar Oliveira

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1 The intermediate value theorem

Theorem 1 (IVT). *If f is a continuous function on a closed interval $[a, b]$, and y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.*

1. Draw a picture that explains the statement above and one that shows how it fails if f is not continuous.

2. The IVT is a great tool to show that solutions to some equations exist, even if we can not find them explicitly. Use it to prove that the equation

$$\sqrt{2x + 5} = 4 - x^2$$

has a solution.

3. Explain why the equation $\cos x = x$ has at least one solution.

4. **True or false?** Suppose that during half-time at a basketball game the score of the home team was 36 points. There had to be at least one moment in the first half when the home team had exactly 25 points.

2 Continuous extension of a function

Sometimes the formula that describes a function f does not make sense at a point $x = c$. It might nevertheless be possible to enlarge the domain of f, creating a new function that is continuous at $x = c$.

For example, the function $f(x) = \frac{\sin x}{x}$ is continuous everywhere but at 0 since it is not in the domain of f. However, we can define a new function

$$F(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}$$

that is continuous at **every** real number and coincides with the old f on f's domain. Which number a makes this work? ______ (1). We say that F is a **continuous extension** of f.

1. Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \quad x \neq 2$$

has a continuous extension to $x = 2$, and find that extension.

3 Limits involving infinity

We write

$$\lim_{x\to\infty} f(x) = L$$

if, as x moves increasingly far from 0 in the positive direction, $f(x)$ gets arbitrarily close to L. Similarly,

$$\lim_{x\to-\infty} f(x) = L$$

if, as x moves increasingly far from 0 in the negative direction, $f(x)$ gets arbitrarily close to L.

Good news! This concept comes with some tools that we have already seen:

> **Theorem 2.** *All the Limit Laws that we saw for ordinary limits are true when we replace* $\lim_{x\to c}$ *by* $\lim_{x\to\infty}$ *or* $\lim_{x\to-\infty}$*. In other words, the variable* x *may approach a finite number* c *or* $\pm\infty$*.*

1. Compute the following limits:

   (a) $\displaystyle\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2}$.

   (b) $\displaystyle\lim_{x\to-\infty} \frac{11x+2}{2x^3-1}$.

> **Definition 3.** A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either
>
> $$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

1. Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3-2}{|x|^3+1}.$$