Derivatives

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1 Definition, interpretation and more

Definition 1. The **derivative of a function** f **at a point** x_0, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{1}$$

provided this limit exists.

Remark 2. The following are all interpretations for the limit (1):

(a) The slope of the graph of $y = f(x)$ at $x = x_0$.

(b) The slope of the tangent to the curve $y = f(x)$ at $x = x_0$.

(c) The rate of change of $f(x)$ with respect to x at $x = x_0$.

(d) The derivative $f'(x_0)$.

The process of calculating a derivative is called **differentiation**. Some of the notations for the derivative of $y = f(x)$ that you can find in textbooks are

$$f'(x), \quad \frac{df}{dx}(x), \quad \frac{d}{dx}f(x), \quad y', \quad D(f)(x).$$

We will stick to the first two.

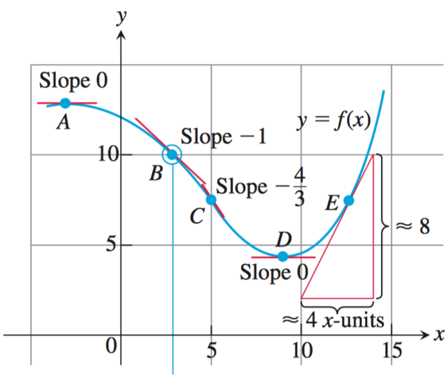
1. Find the derivative of $f(x) = \dfrac{x}{x-1}$ for $x > 0$.

There is an alternative formula for the derivative:

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

1. Using the alternative formula above, find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

We can often make a reasonable plot of the derivative of $y = f(x)$ by estimating the slopes on the graph of f. Look at the graph of f below and sketch the one of f'. After you do that, write down what you can learn about f from the graph of f' alone.



2 What can go wrong with f that will impact f'?

Of course the limit (1) may fail to exist. Similar to what we saw before, it could be the case that a function only has *side derivatives* at a point:

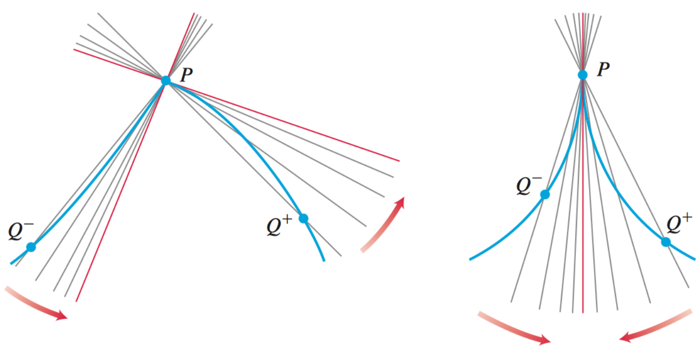
$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}, \quad \textbf{Right-hand derivative at } a$$

$$\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}, \quad \textbf{Left-hand derivative at } a$$

If the limits above are not the same, the function is not differentiable at a. The most popular example of this is the function $f(x) = |x|$.

1. Show that the function $f(x) = |x|$ is not differentiable at 0.

More generally, the picture below shows that *corners* and *cusps* in the graph prevent a function from having a derivative at these points.



1. a *corner*, where the one-sided derivatives differ.

2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.

An important fact about derivatives is the following:

Theorem 3. *If f has a derivative at a point $x = c$, then f is continuous at $x = c$.*

This is very useful when checking whether a function has or not a derivative by looking at its graph: if you see any discontinuity, the derivative does not exist there!

Another important fact:

If the tangent line to the graph of f at a point is parallel to the y-axis, f does not have a derivative at this point.

This is because a vertical line (that is the same as a line parallel to the y-axis) has "∞ slope", therefore the limit in (1) is not a number.

1. For each one of the four graphs below, determine the points at which the functions do not have a derivative. Determine also if they are continuous at these points.