Derivatives of inverse functions and log   Name: ____________

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1 Introduction and examples

Consider the function $f(x) = (1/2)x + 1$ and its inverse $f^{-1}(x) = 2x - 2$. Computing derivatives:

$$f'(x) = 1/2, \quad (f^{-1})' = 2.$$

The derivatives are reciprocals of one another, so the slopes of f and f^{-1} are reciprocal as well. **This is not a special case**, and the next theorem tells us that formally.

Theorem 1. *If f is a one-to-one differentiable function with inverse f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and*

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

This theorem makes two assertions:

1. The conditions under which f^{-1} is differentiable at a.
2. A formula to compute the derivative of f^{-1} when it exists.

Example: find the derivative of f^{-1} for $f(x) = x^2$, $x > 0$.

Example: Let $f(x) = 2x + \cos x$. Find $(f^{-1})'$.

2 Derivative of the natural logarithm

1. Since we know the exponential function $f(x) = e^x$ is differentiable everywhere, we can apply the previous theorem to find the derivative of its inverse $f^{-1}(x) = \ln x$. Show that $(f^{-1})'(x) = \frac{1}{x}$.

2. Computational example: compute the derivative of $y = \sqrt{\ln \sqrt{t}}$.

3. Logarithms and exponentials can come with different bases: if $a > 0$, compute the derivative of $y = a^x$. If $a > 0, a \neq 1$, differentiate $y = \log_a x$.

3 Logarithm differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This process is called **logarithm differentiation** and we illustrate it with an example:

1. Find dy/dx if

$$y = \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}, \quad x > 1.$$

2. Differentiate $f(x) = x^x$, $x > 0$.