Math 2940: Symmetric matrices have real eigenvalues

The Spectral Theorem states that if A is an $n \times n$ symmetric matrix with real entries, then it has n orthogonal eigenvectors. The first step of the proof is to show that all the roots of the characteristic polynomial of A (i.e. the eigenvalues of A) are real numbers.

Recall that if $z = a + bi$ is a complex number, its complex conjugate is defined by $\overline{z} = a - bi$. We have $z\overline{z} = (a + bi)(a - bi) = a^2 + b^2$, so $z\overline{z}$ is always a nonnegative real number (and equals 0 only when $z = 0$). It is also true that if w, z are complex numbers, then $\overline{wz} = \overline{w}\,\overline{z}$.

Let $\mathbf{v}$ be a vector whose entries are allowed to be complex. It is no longer true that $\mathbf{v} \cdot \mathbf{v} \geq 0$ with equality only when $\mathbf{v} = \mathbf{0}$. For example,

$$\begin{bmatrix} 1 \\ i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + i^2 = 0.$$

However, if $\overline{\mathbf{v}}$ is the complex conjugate of $\mathbf{v}$, it is true that $\overline{\mathbf{v}} \cdot \mathbf{v} \geq 0$ with equality only when $\mathbf{v} = \mathbf{0}$. Indeed,

$$\begin{bmatrix} a_1 - b_1 i \\ a_2 - b_2 i \\ \vdots \\ a_n - b_n i \end{bmatrix} \cdot \begin{bmatrix} a_1 + b_1 i \\ a_2 + b_2 i \\ \vdots \\ a_n + b_n i \end{bmatrix} = (a_1^2 + b_1^2) + (a_2^2 + b_2^2) + \cdots + (a_n^2 + b_n^2)$$

which is always nonnegative and equals zero only when all the entries a_i and b_i are zero.

With this in mind, suppose that λ is a (possibly complex) eigenvalue of the real symmetric matrix A. Thus there is a nonzero vector $\mathbf{v}$, also with complex entries, such that $A\mathbf{v} = \lambda\mathbf{v}$. By taking the complex conjugate of both sides, and noting that $\overline{A} = A$ since A has real entries, we get $\overline{A\mathbf{v}} = \overline{\lambda\mathbf{v}} \Rightarrow A\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$. Then, using that $A^T = A$,

$$\begin{aligned} \overline{\mathbf{v}}^T A\mathbf{v} &= \overline{\mathbf{v}}^T(A\mathbf{v}) = \overline{\mathbf{v}}^T(\lambda\mathbf{v}) = \lambda(\overline{\mathbf{v}} \cdot \mathbf{v}), \\ \overline{\mathbf{v}}^T A\mathbf{v} &= (A\overline{\mathbf{v}})^T\mathbf{v} = (\overline{\lambda}\overline{\mathbf{v}})^T\mathbf{v} = \overline{\lambda}(\overline{\mathbf{v}} \cdot \mathbf{v}). \end{aligned}$$

Since $\mathbf{v} \neq \mathbf{0}$, we have $\overline{\mathbf{v}} \cdot \mathbf{v} \neq 0$. Thus $\lambda = \overline{\lambda}$, which means $\lambda \in \mathbf{R}$.