Textbook Exercise:

1.1. No. Consider the following:
\[ P(X_{n+1} = 0 \mid X_n = 1, X_{n-1} = 0) = 0 \]
\[ P(X_{n+1} = 0 \mid X_n = 1, X_{n-1} = 2) = \frac{1}{2} \]
\[ P(X_{n+1} = 0 \mid X_n = 1, X_{n-1} = 0) \neq \]
\[ P(X_{n+1} = 0 \mid X_n = 1, X_{n-1} = 2) \]
Which violates Markov Property.

\[ \square \]

1.2.
\[
P = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0
\frac{1}{25} & \frac{8}{25} & \frac{16}{25} & 0 & 0 & 0
0 & \frac{4}{25} & \frac{12}{25} & \frac{9}{25} & 0 & 0
0 & 0 & \frac{9}{25} & \frac{12}{25} & \frac{9}{25} & 0
0 & 0 & 0 & \frac{6}{25} & \frac{8}{25} & \frac{12}{25}
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[ \square \]

1.5.
(a) \[ P(X_4 = 4 \mid X_1 = 1) = P^3(1, 4) = 0.4^3 = 0.064. \]
(b) \[ P(X_4 = 0 \mid X_1 = 1) = P^3(1, 0) = 0.4 \cdot 0.6 \cdot 0.6 + 0.6 \cdot 1.1 = 0.744. \]

\[ \square \]
1.6.
(a) \[ P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix} \]

(b). \[ P(X_2 = A \mid X_0 = A) = P^2(A, A) = \frac{3}{4} \]
\[ P(X_2 = B \mid X_0 = A) = P^2(A, B) = \frac{1}{8} \]
\[ P(X_2 = C \mid X_0 = A) = P^2(A, C) = \frac{1}{8} \]
\[ P(X_3 = B \mid X_0 = A) = P^2(A, B) = \frac{13}{32} \]

1.7.
(a) \[ P = \begin{pmatrix} RR & RS & SR & SS \\ RR & 0.6 & 0.4 & 0 & 0 \\ RS & 0 & 0 & 0.6 & 0.4 \\ SR & 0.6 & 0.4 & 0 & 0 \\ SS & 0 & 0 & 0.3 & 0.7 \end{pmatrix} \]

(b) \[ P^2 = \begin{pmatrix} 0.36 & 0.24 & 0.24 & 0.16 \\ 0.36 & 0.24 & 0.12 & 0.28 \\ 0.36 & 0.24 & 0.24 & 0.16 \\ 0.18 & 0.12 & 0.21 & 0.49 \end{pmatrix} \]

(c). Two cases satisfy:

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<th>Sunday</th>
<th>Mon</th>
<th>Tues</th>
<th>Wed</th>
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<tbody>
<tr>
<td>①</td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>R</td>
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<tr>
<td>②</td>
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Thus \[ P \text{ (Rainy Wed \mid Sunny Sunday \& Mon)} = P^2(S_S, S_R) + P^2(S_S, R_R) = 0.39 \]
Additional Problems:

1. (a).
\[ P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0.4 & 0 & 0.6 & 0 & 0 & 0 \\
0 & 0.4 & 0 & 0.6 & 0 & 0 \\
0 & 0 & 0.4 & 0 & 0.6 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \]

(b).
\[ \lim_{n \to \infty} P^n = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0.58 & 0 & 0 & 0 & 0 & 0.42 \\
0.31 & 0 & 0 & 0 & 0 & 0.69 \\
0.12 & 0 & 0 & 0 & 0 & 0.88 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \]

(c). If the gambler starts with $1, in the long run he will end up with winning $4 with probability 0.42.

2. (a). (b).
\[
\begin{bmatrix}
0.3 & 0.6 & 0.1
\end{bmatrix}
\begin{pmatrix}
0.7 & 0.2 & 0.1 \\
0.3 & 0.5 & 0.2 \\
0.2 & 0.4 & 0.4
\end{pmatrix}
= \begin{bmatrix}
0.41 & 0.4 & 0.19
\end{bmatrix}.
\]
3. Base case:

\[ m = 1 \quad P(X_{n+1} = j \mid X_n = i) = P(i, j) \quad \text{by definition.} \]

Induction Hypothesis:

Suppose \( m = k \) \( P(X_{n+k} = j \mid X_n = i) = P^k(i, j) \) holds.

then

\[ P(X_{n+k+1} = j \mid X_n = i) \]

\[ = \sum_u P(X_{n+k+1} = j, X_{n+k} = u \mid X_n = i) \]

\[ = \sum_u P(X_{n+k+1} = j \mid X_{n+k} = u, X_n = i) P(X_{n+k} = u \mid X_n = i) \quad (\star) \]

\[ = \sum_u P(X_{n+k+1} = j \mid X_{n+k} = u) P(X_{n+k} = u \mid X_n = i) \quad (\star\star) \]

\[ = \sum_u P^k(u, j) P^k(i, u) \]

\[ = P^{k+1}(i, j) \quad (\Delta) \]

(\star) follows from rules of conditional probability.

(\star\star) follows from Markov property.

(\Delta) follows from matrix multiplication.