Textbook Exercise:

1.8. To identify recurrent / transient / irreducible states / sets of states, the most straightforward tool will be a diagram:

(a) Recurrent: \( \{2, 4\} \)
Transient: \( \{1, 3, 5\} \)
closed irreducible: \( \{2, 4\} \)

(b) Recurrent: \( \{1, 4, 5, 6\} \)
Transient: \( \{2, 3\} \)
closed irreducible: \( \{1, 4, 5, 6\} \)

(c) Recurrent: \( \{2, 4, 1, 5\} \)
Transient: \( \{3\} \)
closed irreducible: \( \{2, 4\} \)

(d) Recurrent: \( \{1, 4, 2, 5\} \)
Transient: \( \{3, 6\} \)
closed irreducible: \( \{1, 4\}, \{2, 5\} \)
1.30. (a) \( \left( \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \right) \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.1 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}^2 = \begin{pmatrix} 0.393 & 0.31 & 0.297 \\ \uparrow & \uparrow & \uparrow \\ L & C & G \end{pmatrix} \)

(b) \( \left( \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \right) \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.1 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}^7 = \begin{pmatrix} 0.3947 & 0.3070 & 0.2982 \end{pmatrix} \)

(c). To solve for stationary distribution \( \pi \):

\[
\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \begin{pmatrix} 0.5 & 0.2 & 0.3 & 1 \\ 0.4 & 0.5 & 0.1 & 1 \\ 0.25 & 0.25 & 0.5 & 1 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & 1 \end{pmatrix}
\]

Row Reduction

\[
\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \begin{pmatrix} -0.5 & 0.2 & 1 \\ 0.4 & -0.5 & 1 \\ 0.25 & 0.25 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}
\]

\[
\Rightarrow \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -0.5 & 0.2 & 1 \\ 0.4 & -0.5 & 1 \\ 0.25 & 0.25 & 1 \end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix} 0.3947 & 0.3070 & 0.2982 \end{pmatrix}.
\]
1.37. (a) The transition matrix is given by:

\[
P = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0.8 & 0.2 \\
0 & 0.8 & 0.2 & 0 \\
0.8 & 0.2 & 0 & 0
\end{pmatrix}
\]

Reason:
(for example) If \( X_n = 0 \), the individual has no umbrella at the current location, then the other location must exist 3 umbrellas, and with probability 1 she will have access to 3 umbrellas at time \( n+1 \).

(b) \( \begin{pmatrix} \Pi_1, \Pi_2, \Pi_3, \Pi_4 \end{pmatrix} \left( \begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0.8 & 0.2 \\
0 & 0.8 & 0.2 & 0 \\
0.8 & 0.2 & 0 & 0
\end{array} \right) = \begin{pmatrix} \Pi_1, \Pi_2, \Pi_3, \Pi_4, 1 \end{pmatrix} \)

\[= \begin{pmatrix} \Pi_1, \Pi_2, \Pi_3, \Pi_4 \end{pmatrix} = \begin{pmatrix} \frac{4}{19}, \frac{5}{19}, \frac{5}{19}, \frac{5}{19} \end{pmatrix} \]

So \( \Pi \) (getting wet) = \( \Pi \) (she has no umbrella, and it's raining)

\[= \frac{4}{19} \cdot 0.2 \]

\[= 0.042 \]

\( \square \).
Additional Problems:

1. (a). \[ T = \min \{ n > 1 : X_{n-1} = A, X_n = C \} \]

(b). \[ T = T_A^2 \]

(c). No such stopping time exists.

Combining case 1 and 3, it violates the "deterministic" property of a stopping time.

2. Proof.

There are only two cases: either \( X \) communicates with \( Y \) or not.

(i) Suppose \( X \) communicates with \( Y \).

Then by lemma 1.9, \( Y \) is also recurrent.

Since \( X \) is recurrent, \( p_{yx} = 1 \) by lemma 1.6. It follows that since \( Y \) is recurrent, \( p_{xy} = 1 \) by lemma 1.6 again.

(ii) Suppose \( X \) does not communicate with \( Y \).

Then \( p_{xy} = 0 \) by definition.

\( \square \).
3. (a) \[
E_X [N(X)] = \sum_{k=1}^{\infty} P_X (N(X) > k)
\]
\[
= \sum_{k=1}^{\infty} P_{xx}^k
\]
\[
= \left\{ \begin{array}{ll}
\frac{P_{xx}}{1-P_{xx}} & \text{if } P_{xx} < 1 \text{ (transient)} \\
\infty & \text{if } P_{xx} = 1 \text{ (recurrent)}
\end{array} \right.
\]

(b) By definition of indicator function, \(\sum_{n=1}^{\infty} 1_{\{X_n=x\}}\) is the total counts when state \(x\) is visited, hence it represents the number of visits to \(x\), i.e. \(N(x)\).

(c) \[
E_X [\sum_{n=1}^{\infty} 1_{\{X_n=x\}}] = P (X_n=x | X_0=x) = P^n(x,x).
\]

(d) Proof.
\[
\text{if recurrent } \iff P_{xx}=1 \iff E_X [N(X)] = \infty
\]

\[
(\Rightarrow) E_X [\sum_{n=1}^{\infty} 1_{\{X_n=x\}}] = \infty
\]

\(\text{MCT}\)
\[
(\Rightarrow) \sum_{n=1}^{\infty} E_X [1_{\{X_n=x\}}] = \infty
\]

(Monotonic Convergence Thm)

\[
(\Rightarrow) \sum_{n=1}^{\infty} P^n(x,x) = \infty
\]