Textbook Exercises:

1.14. A finite-state Markov Chain would converge if:

   (i) it's irreducible
   (ii) it's aperiodic.

(a). No, because each state has period 2.

(Also, notice it's an alternating transition matrix.)

(b). Yes, it's irreducible & aperiodic.

(c). No. Each state has period 3.

1.36 The transition matrix:

\[
P = \begin{bmatrix}
0 & 0 & 1 \\
0.05 & 0.95 & 0 \\
0 & 0.02 & 0.98
\end{bmatrix}
\]

(a) \[ \pi^* = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix}
-1 & 0 & 1 \\
0.05 & -0.05 & 1 \\
0 & 0.02 & 1
\end{pmatrix}^{-1} = \begin{pmatrix} 1/71 \ 20/71 \ 50/71 \end{pmatrix}. \]

Therefore the longrun fraction of time spent

with 1 bulb is \[ \frac{20}{71}. \]

(b) \[ E_0(T_0) = \frac{1}{\pi(0)} = \frac{71}{71}. \]
1.48 (a). The transition matrix is given by:

$$
P = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\frac{1}{2} & \frac{1}{2} & & & & & & & & & \\
\frac{1}{2} & \frac{1}{2} & & & & & & & & & \\
\frac{1}{2} & \frac{1}{2} & & & & & & & & & \\
\frac{1}{2} & \frac{1}{2} & & & & & & & & & \\
\frac{1}{2} & \frac{1}{2} & & & & & & & & & \\
\frac{1}{2} & \frac{1}{2} & & & & & & & & & \\
\frac{1}{2} & \frac{1}{2} & & & & & & & & & \\
\frac{1}{2} & \frac{1}{2} & & & & & & & & & \\
\frac{1}{2} & \frac{1}{2} & & & & & & & & & \\
\frac{1}{2} & \frac{1}{2} & & & & & & & & & \\
\frac{1}{2} & \frac{1}{2} & & & & & & & & & \\
\end{bmatrix}
$$

First, notice that this is a doubly stochastic chain, (the rows and columns of $P$ both sum to 1).

By Thm 1.24, $\pi(x) = \frac{1}{N}$ for $x \in \mathcal{X}$. ($N = \# states$).

So $E_{\pi}(T_x) = \frac{1}{\pi(x)} = \frac{1}{\frac{1}{N}} = N = 12$.
Additional Problems:

1. (a). $\Pi = \left( \frac{1}{3}, \frac{2}{3} \right)$

(b). $P_b(T_a = k) = P(X_k = a, X_{k-1} = b, X_{k-2} = b, \ldots, X_1 = b \mid X_0 = b)$

\[= P(X_k = a \mid X_{k-1} = b) \cdot P(X_{k-1} = b \mid X_{k-2} = b) \cdot \ldots \cdot P(X_1 = b \mid X_0 = b)\]

\[= 0.3 \cdot 0.7^{k-1}\]

(c). $E_b(T_a) = \sum_{k=1}^{b_0} P_b(T_a \geq k)$

\[= \sum_{k=1}^{b_0} 0.7^{k-1}\]

\[= \frac{1}{1-0.7}\]

\[= \frac{10}{3}\]

(d). Starting from state $a$ ($X_0 = a$), it will stay at $a$ w/ prob. $0.4$ ($X_1 = a$) or it will go to $b$ w/ prob. $0.6$. So $E_a(T_a) = 0.4 \cdot 1 + 0.6 \cdot (E_b(T_a) + 1)$

\[= 3\]

$\Pi(a) = \frac{1}{3} = \frac{1}{E_a(T_a)}$. √
2. (a). Starting from state \( y \), it has two possibilities:

- it will never visit state \( x \) \( (N(x) = 0) \) or
- it will visit \( x \) at some time point \( k \).

In the latter case, once it's at state \( x \), the number of visits to \( x \) after time \( k \) has the same distribution as \( N(x) \) for the chain started at \( x \), by strong Markov property.

So
\[
\mathbb{E}_y[N(x)] = 0 \cdot (1 - p_{yx}) + p_{yx} \cdot (1 + \mathbb{E}_x[N(x)])
\]

(b) Suppose \( x \) is transient, then \( \mathbb{E}_x[N(x)] < \infty \).

So
\[
\mathbb{E}_y[N(x)] = p_{yx} \left( 1 + \mathbb{E}_x[N(x)] \right) < \infty
\]

Also recall that
\[
\mathbb{E}_y[N(x)] = \sum_{n=1}^{\infty} P^n(y,x) < \infty.
\]

So by test for divergence,
\[
\lim_{n \to \infty} P^n(y,x) = 0.
\]

(c). Since
\[
\lim_{n \to \infty} (tP^n)(x) = \pi(x), \quad \text{we have}
\]
\[
\lim_{n \to \infty} \sum_{y \in \mathbb{X}} \pi(y) P^n(y,x) = \sum_{y \in \mathbb{X}} \lim_{n \to \infty} \pi(y) P^n(y,x)
\]
\[
= \sum_{y \in \mathbb{X}} \pi(y) \lim_{n \to \infty} P^n(y,x)
\]
\[
= \sum_{y \in \mathbb{X}} \left( \pi(y) \cdot 0 \right)
\]
\[
= 0.
\]
\[
= \pi(x).
\]

\( \square \).
3. Notice that it's an irreducible Markov Chain with finite state space, we can apply the "main convergence theorem" such that \( \tau(0) = \frac{1}{E_0(T(0))} \)

Starting from state 0, it will return to 0 in 3 steps (\( 0 \rightarrow -1 \rightarrow -2 \rightarrow 0 \)) or 4 steps (\( 0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 0 \)), each with probability \( \frac{1}{2} \). Hence

\[
E_0(T_0) = \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 4 = \frac{3}{2}
\]

Therefore \( \tau(0) = \frac{1}{\frac{3}{2}} = \frac{2}{3} \).

\( \Box \).