Textbook Exercises:

1.48 b) Denote \( V_x = \min \{ n > 0 : X_n = x \} \).

\[
\mathbb{P}(V_{12} < V_1 \mid X_2 = 2) \mathbb{P}(X_1 = 2) + \mathbb{P}(V_{2} < V_1 \mid X_2 = 12) \mathbb{P}(X_2 = 12).
\]

Notice that if we suppose \( X_1 = 2 \), then \( \mathbb{P}(V_{12} < V_1) = \frac{1}{n-1} \)

\[ = \frac{1}{11} \text{, by the formula found in Gambler's ruin problem where} \]

\( p = \frac{1}{2} \text{. By symmetry, } \mathbb{P}(V_{2} < V_1 \mid X_2 = 12) = \frac{1}{11} \).

Hence the probability of visiting all states before returning to the starting state is \( \frac{1}{11} \cdot \frac{1}{2} + \frac{1}{11} \cdot \frac{1}{2} = \frac{1}{11} \).

1.74. (a) If \( 0 \) is recurrent then \( \mathbb{P}(T_0 < \infty) = 1 \).

\[
\mathbb{P}(T_0 < \infty) = 1 - \mathbb{P}(T_0 = \infty)
\]

\[ = 1 - \frac{p^n}{i=0} \]

So \( \prod_{i=0}^{\infty} p_i \) will make \( 0 \) recurrent.

(b) (c) To make "0" positive recurrent, we need \( \mathbb{E}_0[T_0] < \infty \). So we find the stationary distribution first:
Denote the stationary distribution \( \Pi = (\Pi_0, \Pi_1, \Pi_2, \ldots) \).

Since the transition matrix \( P = \begin{pmatrix} 1-P_0 & P_0 & 0 & 0 & 0 & \cdots \\ 1-P_1 & 0 & P_1 & 0 & 0 & \cdots \\ 1-P_2 & 0 & 0 & P_2 & 0 & \cdots \\ 1-P_3 & 0 & 0 & 0 & P_3 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \)

we have that \( \Pi P = \Pi \Rightarrow \)

\[
\begin{align*}
\Pi_0 &= \Pi_0 \\
\Pi_1 &= \Pi_0 P_0 \\
\Pi_2 &= \Pi_1 P_1 = P_1 P_0 \Pi_0 \\
\Pi_3 &= \Pi_2 P_2 = P_2 P_1 P_0 \Pi_0 \\
\vdots \\
\Pi_n &= \left( \frac{n-1}{\Pi_k P_k} \right) \Pi_0.
\end{align*}
\]

Also \( \sum_{n=1}^{\infty} \left( \frac{n-1}{\Pi_k P_k} \right) \Pi_0 + \Pi_0 = 1 \Rightarrow \Pi_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{n-1}{\Pi_k P_k}} \)

Therefore, to make \( \Pi \) positive recurrent, we need

\( 1 + \sum_{n=1}^{\infty} \left( \frac{n-1}{\Pi_k P_k} \right) < \infty \), i.e. \( \sum_{n=1}^{\infty} \left( \frac{n-1}{\Pi_k P_k} \right) < \infty \).

The stationary distribution is given by

\[
\begin{align*}
\Pi_0 &= \frac{1}{1 + \sum_{n=1}^{\infty} \frac{n-1}{\Pi_k P_k}} \\
\Pi_n &= \left( \frac{n-1}{\Pi_k P_k} \right) \frac{1}{1 + \sum_{n=1}^{\infty} \frac{n-1}{\Pi_k P_k}} \quad \text{for } n \neq 0.
\end{align*}
\]
\[ M = \text{"Expected number of offspring of one individual"} \]

\[ = \sum_{k=0}^{\infty} P(X_n \geq k) \]

\[ = \sum_{k=1}^{\infty} (1-p)^k \]

\[ = \frac{1-p}{p} \]

Consider 2 cases:

1. \( \mu \leq 1 \implies P \geq \frac{1}{2} \implies P=1 \). Extinction occurs with probability 1.

2. \( \mu > 1 \implies P < \frac{1}{2} \).

\[ P = \arg \min \{ \phi(x) = x \} \]

\[ \text{where } \phi(x) = \sum_{k=0}^{\infty} p(1-p)^k x^k = p \sum_{k=0}^{\infty} (x(1-p))^k = \frac{p}{1-x(1-p)} \]

\[ \implies P = \frac{p}{1-p(1-p)} \]

\[ = \min \left( \frac{1 \pm \sqrt{1-4p(1-p)}}{2(1-p)} \right) \]

\[ = \frac{p}{1-p}. \]

Above all, \( P = \begin{cases} 1 & \text{if } P \geq \frac{1}{2} \\ \frac{p}{1-p} & \text{if } P < \frac{1}{2}. \end{cases} \)