Textbook Exercises:

2.18. To find \( P(X = 1) \), suppose:

1. \( X \sim \text{Binomial} \ (n = 20, \ p = 0.1) \)
   \[ P(X = 1) = \binom{20}{1} 0.1^1 \cdot 0.9^{20-1} = 0.27017 \]

2. \( X \sim \text{Poisson} \ (\lambda = \lambda p = 2.0) \)
   \[ P(X = 1) = e^{-2} \cdot \frac{2^1}{1!} = 0.27067 \]

Comparing this result to the one from 1, we see that Poisson is not a bad approximation to Binomial.

2.22

We know that \( N(t) \sim \text{Poisson} (3t) \)
\[ T_n \sim \text{Gamma}(n, 3), \ \mbox{and} \ \ m_n \sim \text{Exponential}(3) \]

(a). \( E(T_{12}) = \frac{12}{3} = 4 \).

(b). By the memoryless property of \( m_n \),
\[ E[T_{12} | N(12) = 5] = E\left[\sum_{i=1}^{12} t_i | N(12) = 5\right] \]
\[ = E\left[\sum_{i=1}^{12-5} t_i\right] + 2 \quad \left( \text{technically,} \ E\left[\sum_{i=1}^{12} t_i\right] + 2 \right) \]
\[ = 7 \cdot \frac{1}{3} + 2 \]
\[ = \frac{13}{3} \]
\[
E[N(5) | N(2) = 5] \\
= E[N(5) - N(2) + N(2) - N(0) | N(2) - N(0) = 5] \\
= E[N(5) - N(2)] + E[N(2) - N(0) | N(2) - N(0) = 5] \text{ (independent increments)} \\
= E[N(5)] - E[N(2)] + 5 \\
= 3.5 - 2.3 + 5 \\
= 14
\]

2.34

Denote the weight of i-th trout by \( Y_i \), the number of trouts caught by time \( t \) by \( N(t) \). Then

\[ N(t) \sim \text{Poisson}(2t). \]

In 2 hours, Edwin is expected to catch \( E(N(2)) = 6 \) trouts.

So

\[
E\left( \sum_{i=1}^{N(2)} Y_i \right) = E[N(2)] E[Y_i] \\
= 6 \cdot 4, \\
= 24
\]

\[
\text{Var}\left( \sum_{i=1}^{N(2)} Y_i \right) = E[N(2)] \text{Var}(Y_i) + \text{Var}(N(2)) E(Y_i)^2 \\
= 6 \cdot 2^2 + 6 \cdot 4^2 \\
= 120
\]

\[
\text{sd}\left( \sum_{i=1}^{N(2)} Y_i \right) = \sqrt{120} = 2\sqrt{30}.
\]
Denote the number of trucks passed by time \( t \) as \( N_T(t) \)

the number of cars passed by time \( t \) as \( N_c(t) \).

\[
N_T(t) \sim \text{Poisson} \left( \frac{2}{3} \cdot 0.1 \cdot t \right) \quad N_c(t) \sim \text{Poisson} \left( \frac{2}{3} \cdot 0.9 \cdot t \right) \quad N_c(t) \perp N_T(t).
\]

(a) \[
P( N_T(60) \geq 1 )
= 1 - P( N_T(60) = 0 )
= 1 - e^{-4 \cdot \frac{4^0}{0!}}
= 1 - e^{-4}
\]

(b) \[
\mathbb{E} \left[ N_T(60) + N_c(60) \mid N_T(60) = 10 \right]
= \mathbb{E} [ N_c(60) ] + 10
= \frac{2}{3} \cdot 0.9 \cdot 60 + 10
= 46
\]

(c) \[
P( N_T(60) = 5, N_c(60) = 45 \mid N_T(60) + N_c(60) = 50)
= \binom{50}{5} 0.1^5 0.9^{45}
= 0.1849
\]
(a) \[ P \left( N(3) = 4 \mid N(1) = 1 \right) \]
\[ = P \left( N(3) - N(1) + N(1) - N(0) = 4 \mid N(1) - N(0) = 1 \right) \]
\[ = P \left( N(3) - N(1) = 4 - 1 \right) \]
\[ = P \left( N(2) = 3 \right) \]
\[ = e^{-2.2} \cdot \frac{(2.2)^3}{3!} \]
\[ = 0.195 \]

(b) \[ P \left( N(1) = 1 \mid N(3) = 4 \right) \]
\[ = \binom{4}{1} \left( \frac{1}{3} \right)^1 \left( \frac{2}{3} \right)^3 \]
\[ = 0.395 \]

Additional Problems:

1. A Poisson Process \( N(t) \) with rate \( \lambda \) is a stochastic process constructed by a sequence of independent, continuously occurring events where \( N(t) \sim \text{Poisson}(\lambda t) \) and \( N(t) \)'s are related via \( \text{P}(N(t) - N(s)) \sim \text{Poisson}(\lambda(t-s)) \) for all \( t > s \) and \( \Theta \) independent increments; whilst a Poisson random variable \( N(s) \) is just a random variable, the value of which follows a probability measure as in Poisson distribution.
For any point in the 4-dimensional cube of side length $t$, 
2. $\sqrt{t}$, the ordering of $(t_1, t_2, t_3, t_4)$ has $4! = 24$ possibilities with equal probability, for example

$$P(t_1 < t_2 < t_3 < t_4) = P(t_2 < t_1 < t_4 < t_3) = \cdots = \frac{1}{24}$$

So the space \( \{(t_1, t_2, t_3, t_4) \in \mathbb{R}^4 : 0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \leq t_3 \} \)

has the same volume as \( \{(t_1, t_2, t_3, t_4) \in \mathbb{R}^4 : 0 \leq t_2 \leq t_1 \leq t_4 \leq t_3 \leq t_3 \} \)

and all other 22 subspaces.

By symmetry, the volume of any such subspace is \( \frac{t^4}{24} \).

(b) If the given joint density is correct, then

$$\int_A f(t_1, t_2, t_3, t_4) dt_1 dt_2 dt_3 dt_4$$

$$= \int_A \lambda^4 e^{-\lambda t} dt_1 dt_2 dt_3 dt_4$$

$$= \lambda^4 e^{-\lambda t} \cdot \text{vol}(A)$$

$$= e^{-\lambda t} \left( \frac{\lambda t)^4}{4!} \right)$$

However, the integral should be equal to 1 since it's integrating over a density function. The problem with the given joint density is that it needs to be divided by $P(N(t)=4)$ because we are conditioning the probability on $N(t)=4$. So the correct joint density is given by:

$$f(t_1, t_2, t_3, t_4) = \frac{\lambda^4 e^{-\lambda t}}{e^{-\lambda t} \left( \frac{\lambda t)^4}{4!} \right)} = \frac{4!}{t^4}$$
(c) Since \( U_i \sim \text{uniform}\ [0, t] \), we can view \((U_1, U_2, U_3, U_4)\) as a random point within the 4-dimensional cube of side length 4 as described in (a). Therefore the joint density would be 
\[
\frac{1}{\text{vol}(A)} = \frac{4!}{t^4}
\]
And we know that the joint density of \((T_1, T_2, T_3, T_4)\) conditioned on \(N(t) = 4\) is given by \(\frac{4!}{t^4}\) as in part (b), which is the same density function of \(V_i\)'s.

\[\square\]

3. (a)

(i) \(M(0) = N(t_0) - N(t_0) = 0\)

(ii) \(M(t+s) - M(s) = (N(t_0) - N(t_0 - t - s)) - (N(t_0) - N(t_0 - s))\)

\[= N(t_0 - s) - N(t_0 - t - s)\]

\[= N(t_0 - s - t + t) - N(t_0 - t - s)\]

\[\sim \text{Poisson}\ (\lambda t)\]

(iii) Consider \(0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq t_0\)

\(M(s_{i+1}) - M(s_i) = N(t_0) - N(t_0 - S_{i+1}) - N(t_0) + N(t_0 - S_i)\)

\[= N(t_0 - S_i) - N(t_0 - S_{i+1})\]

Since \(0 \leq t_0 - S_n \leq t_0 - S_{n-1} \leq \ldots \leq t_0 - S_2 \leq t_0 - S_1 \leq t_0\), we have \(N(t_0 - S_i) - N(t_0 - S_{i+1})\) are independent for \(i = 1, \ldots, n\) and conclude \(M(s)\) has independent increments too.
(6). By (a) we know that \( \{M(t)\} \) is a backward Poisson process, so \( t-L \) is just the first arrival of \( M(t) \). Also notice that \( L=0 \) if there is no arrival by time \( t \), which occurs with probability

\[
1 - \int_0^t \lambda e^{-\lambda x} \, dx = e^{-\lambda t}.
\]

Therefore,

\[
E(t-L) = \int_0^t x \lambda e^{-\lambda x} \, dx + (t-0) \cdot e^{-\lambda t} = \frac{1}{\lambda} (1 - e^{-\lambda t}).
\]

As \( t \to \infty \), \( e^{-\lambda t} \to 0 \), \( E(t-L) \to \frac{1}{\lambda} \).