Math 4740: Proof of Theorem 1.7

**Theorem.** Let \((X_n)\) be a Markov chain on the state space \(X\) with transition matrix \(P\). If \(C \subseteq X\) is finite, closed, and irreducible, then all states in \(C\) are recurrent.

**Proof.** Fix \(x \in C\). Since \(C\) is irreducible, \(y \to x\) for every \(y \in C\). Therefore, there exists \(m(y) \geq 1\) such that \(P^{m(y)}(y, x) > 0\). Set \(\alpha(y) = P^{m(y)}(y, x)\) and \(\alpha = \min\{\alpha(y): y \in C\}\).

Suppose the Markov chain \((X_n)\) is started from \(x\). We need to prove that the chain returns to \(x\) with probability 1. Let \(T_1 = m(x)\), so that \(P_x(X_{T_1} = x) = \alpha(x)\) and \(P_x(X_{T_1} \neq x) = 1 - \alpha(x) \leq 1 - \alpha\). If \(X_{T_1} = x\), we are done. If not, then \(X_{T_1} \in C\) (since \(C\) is closed). Thus, if we run the chain for another \(m(X_{T_1})\) steps, there will be a probability of \(\alpha(X_{T_1})\) that we reach \(x\) at time \(T_2 = T_1 + m(X_{T_1})\). The diagram below illustrates this process.

![Diagram illustrating the proof](image)

Formally, we define a sequence of stopping times \(T_1, T_2, \ldots\) by \(T_1 = m(x)\) and for \(k \geq 1\), \(T_{k+1} = T_k + m(X_{T_k})\). Since each failure probability is \(1 - \alpha(X_{T_k}) \leq 1 - \alpha\), it seems intuitively true that for all \(k \geq 1\),

\[
P_x(T_x > T_k) \leq (1 - \alpha)^k, \tag{1}
\]

where \(T_x = \min\{n \geq 1: X_n = x\}\). Taking the limit as \(k \to \infty\) implies that \(P_x(T_x = \infty) = 0\), that is, \(x\) is recurrent. Therefore it remains to prove (1). We argue by induction on \(k\).

**Base case** \(k = 1\): \(P_x(T_x > T_1) \leq P_x(X_{T_1} \neq x) = 1 - P^m(x, x) = \)
$1 - \alpha(x) \leq 1 - \alpha$. Assume now that (1) holds for $k$. We compute:

$$
P_x(T_x > T_{k+1})
\leq P_x(T_x > T_k, X_{T_{k+1}} \neq x)
= \sum_{\substack{y \in C \\{y \neq x\}}} P_x(T_x > T_k, X_{T_k} = y, X_{T_{k+1}} \neq x)
= \sum_{\substack{y \in C \\{y \neq x\}}} P_x(T_x > T_k, X_{T_k} = y) P_x(X_{T_{k+1}} \neq x \mid T_x > T_k, X_{T_k} = y).
$$

Because the conditions $T_x > T_k, X_{T_k} = y$ depend only on the history up to the stopping time $T_k$, the strong Markov property implies that

$$
P_x(X_{T_{k+1}} \neq x \mid T_x > T_k, X_{T_k} = y) = P_y(X_{m(y)} \neq x) = 1 - \alpha(y) \leq 1 - \alpha.
$$

Therefore,

$$
P_x(T_x > T_{k+1}) \leq \sum_{\substack{y \in C \\{y \neq x\}}} P_x(T_x > T_k, X_{T_k} = y)(1 - \alpha)
= (1 - \alpha) P_x(T_x > T_k)
\leq (1 - \alpha)(1 - \alpha)^k = (1 - \alpha)^{k+1},
$$

using the inductive hypothesis on the last line. This completes the proof. \qed