Math 6710, Fall 2016
Final Exam Solutions

1. First, a student pointed out a subtle thing: if $P(X_i = 0) = p > 0$, then

$$\frac{X_1 + \cdots + X_n}{(X_1^2 + \cdots + X_n^2)^{1/2}} \tag{1}$$

evaluates to $0/0$ with probability $p^n > 0$. This is troublesome because a random variable is supposed to be defined under all circumstances and there is no natural choice for the $0/0$. Fortunately, it doesn't matter. Suppose we make two different choices for how to assign the value of (1) when $X_1 = \cdots = X_n = 0$, leading to random variables U_n, U_n' such that $P(U_n \neq U_n') = p^n$. Since $\sigma^2 > 0$ we have $p < 1$, so $P(U_n \neq U_n') \to 0$. For all $x \in \mathbf{R}$,

$$P(U_n \leq x) \leq P(U_n \neq U_n') + P(U_n' \leq x),$$
$$P(U_n' \leq x) \leq P(U_n \neq U_n') + P(U_n \leq x).$$

If $U_n \Rightarrow U$ then $P(U_n \leq x) \to P(U \leq x)$ for all x at which the distribution function of U is continuous. The inequalities above imply that $P(U_n' \leq x) \to P(U \leq x)$ for all such x, implying that $U_n' \Rightarrow U$ also. For our problem, if we make any choice for the value of $0/0$, for example

$$U_n = \begin{cases} \dfrac{X_1 + \cdots + X_n}{(X_1^2 + \cdots + X_n^2)^{1/2}} & \text{if } X_1, \ldots, X_n \text{ are not all zero,} \\ 0 & \text{if } X_1 = \cdots = X_n = 0, \end{cases}$$

and show that $U_n \Rightarrow Z \sim N(0,1)$, then the same weak convergence would hold for any other choice U_n'.

With that preamble, we proceed to the argument. Write $U_n = W_n Y_n$ where

$$W_n = \frac{X_1 + \cdots + X_n}{\sigma\sqrt{n}}, \qquad Y_n = \left(\frac{\sigma^2 n}{X_1^2 + \cdots + X_n^2}\right)^{1/2}$$

and $Y_n = 1$ when $X_1 = \cdots = X_n = 0$. Then $W_n \Rightarrow Z \sim N(0,1)$ by the classical Central Limit Theorem. With probability 1, $X_1^2 + \cdots + X_n^2$ will be eventually positive for all n past a certain point, so

$$\frac{1}{Y_n^2} = \frac{X_1^2 + \cdots + X_n^2}{\sigma^2 n} \quad \text{for sufficiently large } n.$$

The right side converges to 1 a.s. by the classical Strong Law of Large Numbers, so the left side does also. Let $g(x) = 1/\sqrt{x}$; then each $g(1/Y_n^2) = Y_n$ since $0 < Y_n < \infty$. Because g is continuous at $x = 1$,

$$Y_n = g\left(\frac{1}{Y_n^2}\right) \to g(1) = 1 \quad \text{a.s.}$$

We have shown that $W_n \Rightarrow Z$ and $Y_n \to 1$ a.s., which implies $Y_n \Rightarrow 1$. Using Problem 2 on Homework 10, we conclude that $U_n = W_n Y_n \Rightarrow Z$.

2. (a) Let $S_n = |E_n|$ be the number of edges in (V_n, E_n) for $n \geq 2$. Number the potential edges (i.e. pairs of distinct vertices) from 1 to $\binom{n}{2}$ and define the indicator variables $X_{n,k}$ by $X_{n,k} = 1$ if edge k is present in E_n and $X_{n,k} = 0$ if edge k is absent from E_n. For each n, the $X_{n,k}$ are iid with $P(X_{n,k} = 0) = P(X_{n,k} = 1) = \frac{1}{2}$. In addition, $S_n = X_{n,1} + \cdots + X_{n,\binom{n}{2}}$.

To prove a strong law of large numbers for S_n, we mimic the proof of Theorem 8.3 in the notes (Strong Law for bounded fourth moments). Define the centered variables $\overline{X}_{n,k} = X_{n,k} - \frac{1}{2}$, $\bar{S}_n = S_n - \frac{1}{2}\binom{n}{2} = \overline{X}_{n,1} + \cdots + \overline{X}_{n,\binom{n}{2}}$. The argument in the proof of Theorem 8.3 shows that $E[\bar{S}_n^4] \leq C\binom{n}{2}^2$ where $C = 3E[\overline{X}_{n,1}^4] = \frac{3}{16}$. Continuing as in that proof, Chebyshev/Markov's inequality shows that for all $\varepsilon > 0$,

$$P\left(\frac{|\bar{S}_n|}{\binom{n}{2}} > \varepsilon\right) = P\left(\bar{S}_n^4 > \varepsilon^4 \binom{n}{2}^4\right) \leq \frac{C\binom{n}{2}^2}{\varepsilon^4\binom{n}{2}^4} = \frac{C}{\varepsilon^4\binom{n}{2}^2}.$$

As $\binom{n}{2} \geq \frac{1}{2}(n-1)^2$,

$$\sum_{n=2}^{\infty} P\left(\frac{|\bar{S}_n|}{\binom{n}{2}} > \varepsilon\right) \leq \frac{C}{\varepsilon^4} \sum_{n=2}^{\infty} \frac{4}{(n-1)^4} < \infty.$$

The first Borel-Cantelli lemma implies that

$$P\left(\frac{|\bar{S}_n|}{\binom{n}{2}} > \varepsilon \text{ i.o.}\right) = 0.$$

Since this holds for all $\varepsilon > 0$, it follows that

$$\frac{S_n - \frac{1}{2}\binom{n}{2}}{\binom{n}{2}} = \frac{\bar{S}_n}{\binom{n}{2}} \to 0 \quad \text{a.s.}$$

which is a strong law of large numbers for S_n.

For a central limit theorem, note that each S_n has distribution Binomial$(\binom{n}{2}, \frac{1}{2})$, with mean $\frac{1}{2}\binom{n}{2}$ and variance $\frac{1}{4}\binom{n}{2}$. We aim to show that

$$\frac{S_n - \frac{1}{2}\binom{n}{2}}{\frac{1}{2}\sqrt{\binom{n}{2}}} \Rightarrow Z \sim N(0,1). \tag{2}$$

Let $Y_1, Y_2, \ldots$ be iid, $P(Y_i = 0) = P(Y_i = 1) = \frac{1}{2}$, and let $T_k = Y_1 + \cdots + Y_k \sim$ Binomial$(k, \frac{1}{2})$. The classical Central Limit Theorem says that

$$\frac{T_k - \frac{1}{2}k}{\frac{1}{2}\sqrt{k}} \Rightarrow Z. \tag{3}$$

Both (2) and (3) are statements about the convergence of sequences of distribution functions. The sequence of distribution functions in (2) is a subsequence of the one in (3): take $k = \binom{2}{2}, \binom{3}{2}, \binom{4}{2}, \ldots$. Therefore (3) directly implies (2), since weak convergence of a sequence implies weak convergence of every subsequence. This completes the proof.

Alternatively, we could use the Lindeberg-Feller Central Limit Theorem to prove (2) directly. Set

$$U_{n,k} = \frac{\overline{X}_{n,k}}{\frac{1}{2}\sqrt{\binom{n}{2}}}, \qquad W_n = U_{n,1} + \cdots + U_{n,\binom{n}{2}} = \frac{S_n - \frac{1}{2}\binom{n}{2}}{\frac{1}{2}\sqrt{\binom{n}{2}}}.$$

The nth row of the triangular array has $\binom{n}{2}$ entries rather than n entries as in the statement of Lindeberg-Feller in the notes. However, the proof extends without change to the situation where each row has an arbitrary number of entries. Thus, to show that $W_n \Rightarrow Z$ it suffices to check that:

$$\text{(i)} \lim_{n\to\infty} \sum_{k=1}^{\binom{n}{2}} E[U_{n,k}^2] = 1, \qquad \text{(ii)} \lim_{n\to\infty} \sum_{k=1}^{\binom{n}{2}} E[U_{n,k}^2; |U_{n,k}| > \varepsilon] = 0 \text{ for all } \varepsilon > 0.$$

Both conditions are easily verified. For (i),

$$\sum_{k=1}^{\binom{n}{2}} E[U_{n,k}^2] = \frac{\mathrm{Var}(X_{n,1}) + \cdots + \mathrm{Var}(X_{n,\binom{n}{2}})}{\frac{1}{4}\binom{n}{2}} = 1.$$

For (ii), since each $|U_{n,k}| = 1/\sqrt{\binom{n}{2}}$, if $\varepsilon > 0$ is fixed then for large enough n, each $E[U_{n,k}^2; |U_{n,k}| > \varepsilon] = 0$. Thus $W_n \Rightarrow Z$, as desired.

(b) For each $n \geq 3$, number the triples of distinct vertices from 1 to $\binom{n}{3}$. Let $X_{n,k} = 1$ if triangle k is present in the graph and $X_{n,k} = 0$ if triangle k is absent. Thus, $P(X_{n,k} = 1) = \frac{1}{8}$ and $P(X_{n,k} = 0) = \frac{7}{8}$. Since $T_n = X_{n,1} + \cdots + X_{n,\binom{n}{3}}$, we have $E[T_n] = \frac{1}{8}\binom{n}{3}$. Next we compute $\mathrm{Var}(T_n) = E[T_n^2] - E[T_n]^2$:

$$E[T_n^2] = \sum_{1\leq j,k\leq \binom{n}{3}} E[X_{n,j}X_{n,k}], \qquad E[T_n]^2 = \sum_{1\leq j,k\leq \binom{n}{3}} E[X_{n,j}]E[X_{n,k}].$$

(Note that each term in the latter sum is $\frac{1}{64}$, which is consistent with $E[T_n]^2 = \frac{1}{64}\binom{n}{3}^2$.) So,

$$\mathrm{Var}(T_n) = \sum_{1\leq j,k\leq \binom{n}{3}} \Big(E[X_{n,j}X_{n,k}] - E[X_{n,j}]E[X_{n,k}]\Big) = \sum_{1\leq j,k\leq \binom{n}{3}} \left(E[X_{n,j}X_{n,k}] - \frac{1}{64}\right).$$

The random variable $X_{n,j}X_{n,k}$ is 1 if both triangle j and triangle k are present in the graph, and 0 otherwise. There are three cases:

(1) $j = k$, so both triangles are the same. Then $E[X_{n,j}X_{n,k}] - \frac{1}{64} = \frac{1}{8} - \frac{1}{64} = \frac{7}{64}$. There are $\binom{n}{3}$ terms of this type.

(2) The two triangles share a single edge. There are five distinct edges which must be present in the graph in order for $X_{n,j}X_{n,k}$ to equal 1, so $E[X_{n,j}X_{n,k}] - \frac{1}{64} = \frac{1}{32} - \frac{1}{64} = \frac{1}{64}$. To count the number of terms of this type, suppose triangle j has vertices u, v, w. Triangle k could have vertices u, v, x or u, w, x or v, w, x for any x among the $n - 3$ remaining vertices. Therefore, for each j there are $3(n-3)$ choices for k, and the total number of terms is $3(n-3)\binom{n}{3}$.

(3) The two triangles share no edges. Then $X_{n,j}$ and $X_{n,k}$ are independent, so $E[X_{n,j}X_{n,k}] - \frac{1}{64} = E[X_{n,j}]E[X_{n,k}] - \frac{1}{64} = 0$. It does not matter how many of these terms there are, since they contribute zero to $\text{Var}(T_n)$, but we could still count them if we want to. Given triangle j with vertices u, v, w, triangle k could have vertices u, x, y or v, x, y or w, x, y or x, y, z. This gives $3\binom{n-3}{2} + \binom{n-3}{3}$ possibilities, so the total number of terms is $\left[3\binom{n-3}{2} + \binom{n-3}{3}\right]\binom{n}{3}$. Adding up the total number of terms from cases (1),(2),(3) gives

$$\left[1 + 3\binom{n-3}{1} + 3\binom{n-3}{2} + \binom{n-3}{3}\right]\binom{n}{3} = \binom{n}{3}^2$$

which is the correct number of terms in the sum. Again, this is unnecessary but it is a good way to check that the numbers in cases (1) and (2) are right.

From the counting procedure above,

$$\text{Var}(T_n) = \frac{7}{64}\binom{n}{3} + \frac{1}{64} \cdot 3(n-3)\binom{n}{3} \le Cn^4$$

where we can take $C = \frac{1}{128}$. To prove a strong law of large numbers for T_n, use Chebyshev's inequality. For all $\varepsilon > 0$,

$$P\left(\left|\frac{T_n - \frac{1}{8}\binom{n}{3}}{\binom{n}{3}}\right| > \varepsilon\right) \le \frac{Cn^4}{\varepsilon^2\binom{n}{3}^2}.$$

The denominator has order n^6, so the sum over n of these probabilities should be finite. Indeed, since $\binom{n}{3} \sim \frac{n^3}{6}$ as $n \to \infty$ and

$$\sum_{n=3}^{\infty} \frac{Cn^4}{\varepsilon^2(n^3/6)^2} = \frac{36C}{\varepsilon^2}\sum_{n=3}^{\infty}\frac{1}{n^2} < \infty,$$

the limit comparison test implies that

$$\sum_{n=3}^{\infty} P\left(\left|\frac{T_n - \frac{1}{8}\binom{n}{3}}{\binom{n}{3}}\right| > \varepsilon\right) < \infty.$$

By the first Borel-Cantelli lemma,

$$P\left(\left|\frac{T_n - \frac{1}{8}\binom{n}{3}}{\binom{n}{3}}\right| > \varepsilon \text{ i.o.}\right) = 0.$$

Since this holds for all $\varepsilon > 0$, it follows that

$$\frac{T_n - \frac{1}{8}\binom{n}{3}}{\binom{n}{3}} \to 0 \quad \text{a.s.}$$

which is the desired strong law of large numbers for T_n.

3. (a) Define

$$X_{n,1} = \begin{cases} |X| & \text{if } X_n > |X|, \\ X_n & \text{if } |X_n| \le |X|, \\ -|X| & \text{if } X_n < -|X|, \end{cases} \qquad X_{n,2} = X_n - X_{n,1}.$$

Because $|X_n - X| \le |X_{n,1} - X| + |X_n - X_{n,1}| = |X_{n,1} - X| + |X_{n,2}|$, it suffices to prove that $E|X_{n,1} - X| \to 0$ and $E|X_{n,2}| \to 0$. By construction,

$$|X_{n,2}| = \begin{cases} |X_n| - |X| & \text{if } |X_n| > |X|, \\ 0 & \text{if } |X_n| \le |X|. \end{cases}$$

Since $X_n \to X$ a.s., also $|X_n| \to |X|$ a.s. and therefore $|X_{n,2}| \to 0$ a.s. It follows that $X_{n,1} = X_n - X_{n,2} \to X - 0$ a.s., so that $|X_{n,1} - X| \to 0$ a.s. Also, $|X_{n,1}| \le |X|$ and $|X_{n,1} - X| \le 2|X|$. Thus $E|X_{n,1}| \to E|X|$ and $E|X_{n,1} - X| \to 0$ by dominated convergence.

Next, observe that $|X_{n,2}| = |X_n| - |X_{n,1}|$. Therefore, $E|X_{n,2}| = E|X_n| - E|X_{n,1}| \to E|X| - E|X| = 0$. This completes the proof.

(b) Statement: For $p > 1$, if $X_n \to X$ a.s. and $E[|X_n|^p] \to E[|X|^p]$, then $E[|X_n - X|^p] \to 0$.

Proof: Define $X_{n,1}, X_{n,2}$ as in part (a). If we can show that $E[|X_{n,1} - X|^p] \to 0$ and $E[|X_{n,2}|^p] \to 0$, then $\|X_n - X\|_p \le \|X_{n,1} - X\|_p + \|X_{n,2}\|_p \to 0$, which implies that $E[|X_n - X|^p] \to 0$. By the argument in part (a), $X_{n,1} \to X$ a.s., which implies that $|X_{n,1}|^p \to |X|^p$ a.s. and $|X_{n,1} - X|^p \to 0$ a.s. The bounds $|X_{n,1}|^p \le |X|^p$ and $|X_{n,1} - X|^p \le (2|X|)^p$, along with $E[|X|^p] < \infty$, imply that $E[|X_{n,1}|^p] \to E[|X|^p]$ and $E[|X_{n,1} - X|^p] \to 0$ by dominated convergence.

Next, since $|X_{n,1}| + |X_{n,2}| = |X_n|$, we have $|X_{n,1}|^p + |X_{n,2}|^p \le |X_n|^p$. This is a general fact: If $p > 1$ and $a, b \ge 0$, then $a^p + b^p \le (a+b)^p$. Quick proof: The statement is trivial if $a + b = 0$. If $a + b > 0$ then $a/(a+b), b/(a+b) \in [0,1]$ and so

$$\left(\frac{a}{a+b}\right)^p + \left(\frac{b}{a+b}\right)^p \le \frac{a}{a+b} + \frac{b}{a+b} = 1 \implies a^p + b^p \le (a+b)^p.$$

(The fact also follows from convexity of $x \mapsto x^p$.) We obtain $E[|X_{n,2}|^p] \le E[|X_n|^p] - E[|X_{n,1}|^p]$, so

$$\limsup_{n\to\infty} E[|X_{n,2}|^p] \le \limsup_{n\to\infty} E[|X_n|^p] - \liminf_{n\to\infty} E[|X_{n,1}|^p] = E[|X|^p] - E[|X|^p] = 0,$$

meaning that $E[|X_{n,2}|^p] \to 0$. This completes the proof.

4. Our overall strategy is to define random variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ on the same probability space such that the X_i are iid with distribution function F and the Y_i are iid with distribution function U (i.e. $Y_i \sim \text{Uniform}[0,1]$). Let

$$F_n(x) = \tfrac{1}{n} \cdot \#\{1 \le i \le n : X_i \le x\}, \qquad U_n(y) = \tfrac{1}{n} \cdot \#\{1 \le i \le n : Y_i \le y\}$$

and

$$D_n^{(F)} = \sup_{x\in\mathbf{R}} |F_n(x) - F(x)|, \qquad D_n^{(U)} = \sup_{y\in\mathbf{R}} |U_n(y) - U(y)|.$$

In our construction, it will hold that $D_n^{(F)} \le D_n^{(U)}$, with equality when F is continuous. This will prove both parts (a) and (b).

The probability space is $\Omega = (0,1)^n$ with Lebesgue measure. (That is, put Lebesgue measure on each interval $(0,1)$ and take the product measure.) Given $\omega = (\omega_1, \ldots, \omega_n)$, let $Y_i(\omega) = \omega_i$ for $1 \le i \le n$, so the Y_i are iid Uniform$[0,1]$ random variables.

For $0 < y < 1$, define $F^{-1}(y) = \inf\{x \in \mathbf{R} : F(x) \ge y\}$. For each i, let $X_i = F^{-1}(Y_i)$. The X_i are independent, and each X_i has distribution function F by the proof of Theorem 3.2 in the notes. In that proof, it is shown that for all $x \in \mathbf{R}$, $X_i \le x$ if and only if $Y_i \le F(x)$. Hence

$$nF_n(x) = \#\{i : X_i \le x\} = \#\{i : Y_i \le F(x)\} = nU_n(F(x))$$

and so $|F_n(x) - F(x)| = |U_n(F(x)) - F(x)| = |U_n(F(x)) - U(F(x))|$. (The last equality is because $U(y) = y$ for all $0 \le y \le 1$.) It follows that

$$\sup_{x \in \mathbf{R}} |F_n(x) - F(x)| = \sup_{x \in \mathbf{R}} |U_n(F(x)) - U(F(x))| \le \sup_{y \in \mathbf{R}} |U_n(y) - U(y)|.$$

In other words, $D_n^{(F)} \le D_n^{(U)}$, proving part (b).

If F is continuous, the range of F includes every $0 < y < 1$ by the Intermediate Value Theorem. Thus

$$\sup_{x \in \mathbf{R}} |F_n(x) - F(x)| = \sup_{x \in \mathbf{R}} |U_n(F(x)) - U(F(x))| \ge \sup_{0<y<1} |U_n(y) - U(y)|.$$

In fact, $|U_n(y) - U(y)| = 0$ for all $y \notin (0,1)$. If $y \le 0$ then $U(y) = 0$, and also $U_n(y) = 0$ since each $Y_i > 0$. If $y \ge 1$ then $U(y) = 1$, and also $U_n(y) = 1$ since each $Y_i < 1$. Therefore,

$$\sup_{0<y<1} |U_n(y) - U(y)| = \sup_{y \in \mathbf{R}} |U_n(y) - U(y)|$$

so we have shown the reverse inequality $D_n^{(F)} \ge D_n^{(U)}$, proving part (a).

5. (a) From Theorems 18.1 and 18.2 in the notes, $\{S_n\}$ is recurrent if and only if $\sum_{n=0}^{\infty} P(S_n = 0) = \infty$. Since

$$P(R_t = 0) = \sum_{n=0}^{\infty} P(N(t) = n) P(R_t = 0 \mid N(t) = n) = \sum_{n=0}^{\infty} P(N(t) = n) P(S_n = 0),$$

we use Fubini's Theorem to compute

$$\int_0^\infty P(R_t = 0)\, dt = \int_0^\infty \sum_{n=0}^{\infty} P(N(t) = n) P(S_n = 0)\, dt = \sum_{n=0}^{\infty} P(S_n = 0) \int_0^\infty P(N(t) = n)\, dt.$$

For each $n \ge 0$, $\displaystyle\int_0^\infty P(N(t) = n)\, dt$ is the expected amount of time that the Poisson process stays at level n, which is the expected amount of time between the nth and $(n+1)$st arrivals, which equals 1. This argument can be made rigorous, but we can also evaluate the integral directly. Since $N(t) \sim \text{Poisson}(t)$,

$$\int_0^\infty P(N(t) = n)\, dt = \int_0^\infty e^{-t} \left(\frac{t^n}{n!}\right) dt.$$

We prove by induction that this integral equals 1 for every $n \geq 0$. When $n = 0$, $\int_0^\infty e^{-t}\, dt = 1$. When $n \geq 1$, integration by parts gives

$$\int_0^\infty e^{-t} \left(\frac{t^n}{n!}\right) dt = -e^{-t} \left(\frac{t^n}{n!}\right)\Big|_{t=0}^\infty - \int_0^\infty -e^{-t} \left(\frac{t^{n-1}}{(n-1)!}\right) dt = \int_0^\infty e^{-t} \left(\frac{t^{n-1}}{(n-1)!}\right) dt,$$

which equals 1 by the inductive hypothesis. In conclusion,

$$\int_0^\infty P(R_t = 0)\, dt = \sum_{n=0}^\infty P(S_n = 0),$$

which is infinite if and only if $\{S_n\}$ is recurrent.

(b) If $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$, define the norm $\|x\| = \max\{|x_i| : 1 \leq i \leq d\}$. This is just for consistency with the notes; the statements and arguments work for any norm on $\mathbf{R}^d$.

Statement: $\{S_n\}$ is recurrent if and only if $\displaystyle\int_0^\infty P(\|R_t\| < \varepsilon)\, dt = \infty$ for all $\varepsilon > 0$.

Proof: There are two steps:

(1) $\{S_n\}$ is recurrent if and only if $\displaystyle\sum_{n=0}^\infty P(\|S_n\| < \varepsilon) = \infty$ for all $\varepsilon > 0$.

(2) For fixed $\varepsilon > 0$, $\displaystyle\sum_{n=0}^\infty P(\|S_n\| < \varepsilon) = \int_0^\infty P(\|R_t\| < \varepsilon)\, dt$.

Step 2 is proved exactly as in part (a):

$$\begin{aligned}\int_0^\infty P(\|R_t\| < \varepsilon)\, dt &= \int_0^\infty \sum_{n=0}^\infty P(N(t) = n) P(\|S_n\| < \varepsilon)\, dt \\ &= \sum_{n=0}^\infty P(\|S_n\| < \varepsilon) \int_0^\infty P(N(t) = n)\, dt = \sum_{n=0}^\infty P(\|S_n\| < \varepsilon).\end{aligned}$$

Therefore it suffices to prove step 1. One direction is quick. Suppose the sum in step 1 is finite for some $\varepsilon > 0$. Then by the first Borel-Cantelli lemma, $P(\|S_n\| < \varepsilon \text{ i.o.}) = 0$, so 0 is not a recurrent value of $\{S_n\}$. It follows from Theorem 18.1 that $\{S_n\}$ is not recurrent.

Conversely, suppose $\{S_n\}$ is not recurrent. By Theorem 18.1, 0 is not a recurrent value, so there exists $\varepsilon > 0$ such that $P(\|S_n\| < \varepsilon \text{ i.o.}) = 0$. It follows that there exists $\delta > 0$ such that $P(\|S_n\| \geq$

δ for all $n \geq 1) > 0$. To see why, assume for contradiction that for every $\delta > 0$, $P(\|S_n\| \geq \delta$ for all $n \geq 1) = 0$. Let $\tau_0 = 0$ and for $k \geq 1$,

$$\tau_k = \min\{n > \tau_{k-1} : \|S_n - S_{\tau_{k-1}}\| < \varepsilon/2^k\}.$$

We show by induction that each τ_k is a.s. finite. Since τ_{k-1} is an a.s. finite stopping time, the sequence $\{S_{\tau_{k-1}+m} - S_{\tau_{k-1}}\}_{m\geq 0}$ has the same law as $\{S_m\}_{m\geq 0}$ even when we condition on the value of τ_{k-1}. Using the assumption, with probability 1 there exists $n > \tau_{k-1}$ such that $\|S_n - S_{\tau_{k-1}}\| < \varepsilon/2^k$, so τ_k is a.s. finite. For each k,

$$\|S_{\tau_k}\| = \|S_{\tau_k} - S_{\tau_0}\| \leq \sum_{j=1}^{k} \|S_{\tau_j} - S_{\tau_{j-1}}\| < \sum_{j=1}^{k} \frac{\varepsilon}{2^k} < \varepsilon.$$

Hence $P(\|S_n\| < \varepsilon \text{ i.o.}) = 1$, a contradiction.

We have shown that if $\{S_n\}$ is not recurrent, then there exists $\delta > 0$ such that $p := P(\|S_n\| \geq \delta$ for all $n \geq 1) > 0$. Let $V = \#\{n \geq 0 : \|S_n\| < \delta/2\}$. Then

$$E[V] = \sum_{n=0}^{\infty} P(\|S_n\| < \delta/2)$$

and also $E[V] = \sum_{k=0}^{\infty} P(V > k)$. Let $\alpha_0 = 0$ and for $k \geq 1$,

$$\alpha_k = \min\{n > \alpha_{k-1} : \|S_n\| < \delta/2\}.$$

Each α_k is a stopping time, and $P(V > k) = P(\alpha_k < \infty)$. Given that $\alpha_{k-1} < \infty$, and given the history of the random walk up to time α_{k-1}, we know that $\|S_{\alpha_{k-1}}\| < \delta/2$. Also, with probability $p > 0$, $\|S_n - S_{\alpha_{k-1}}\| \geq \delta$ for every $n > \alpha_{k-1}$. This implies that $\|S_n\| \geq \|S_n - S_{\alpha_{k-1}}\| - \|S_{\alpha_{k-1}}\| \geq \delta/2$ for every $n > \alpha_{k-1}$, so $\alpha_k = \infty$. We conclude that $P(\alpha_k < \infty \mid \alpha_{k-1} < \infty) \leq 1 - p$, and so $P(\alpha_k < \infty) \leq (1-p)^k$ by induction. Hence

$$\sum_{n=0}^{\infty} P(\|S_n\| < \delta/2) = E[V] = \sum_{k=0}^{\infty} P(\alpha_k < \infty) \leq \sum_{k=0}^{\infty} (1-p)^k < \infty.$$

This finishes the proof of the converse direction in step 1.

(c) We construct the simple random walk in the following manner. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be the standard basis vectors in $\mathbf{R}^d$ (e.g. $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$). Let $V_1, V_2, \ldots$ be iid, $P(V_i = j) = 1/d$ for $1 \leq j \leq d$. Let $W_1, W_2, \ldots$ be iid and independent of the V_i, $P(W_i = 1) = P(W_i = -1) = 1/2$. Then we can set $X_i = W_i \mathbf{e}_{V_i}$ and $S_n = X_1 + \cdots + X_n$. The jth coordinate of the random walk after n steps is

$$S_n^{(j)} = \sum_{i=1}^{n} \mathbf{1}\{V_i = j\} W_i.$$

Passing to the continuous time version $R_t = S_{N(t)}$, the jth coordinate at time t is

$$R_t^{(j)} = \sum_{i=1}^{N(t)} \mathbf{1}\{V_i = j\} W_i.$$

Fix $t \geq 0$. For each $1 \leq j \leq d$, let $N_j(t) = \#\{1 \leq i \leq N(t) : V_i = j\}$. Problem 5 on Homework 12 (thinning of Poisson variables) shows that each $N_j(t) \sim \text{Poisson}(t/d)$ and that the $N_j(t)$ are independent. Given the values of the $N_j(t)$, the $R_t^{(j)}$ are conditionally independent with

$$P(R_t^{(j)} = x \mid N_1(t) = n_1, \ldots, N_d(t) = n_d) = P(T_{n_j} = x)$$

where $\{T_n\}$ is a discrete time simple random walk on $\mathbf{Z}$. Therefore,

$$\begin{aligned}
&P(R_t^{(1)} = x_1, \ldots, R_t^{(d)} = x_d) \\
&\quad = \sum_{n_1,\ldots,n_d \geq 0} P(N_1(t) = n_1, \ldots, N_d(t) = n_d) P(R_t = (x_1, \ldots, x_d) \mid N_1(t) = n_1, \ldots, N_d(t) = n_d) \\
&\quad = \sum_{n_1,\ldots,n_d \geq 0} \left(\prod_{j=1}^{d} P(N_j(t) = n_j) \right) \left(\prod_{j=1}^{d} P(T_{n_j} = x_j) \right) \\
&\quad = \left(\sum_{n_1=0}^{\infty} P(N_1(t) = n_1) P(T_{n_1} = x_1) \right) \cdots \left(\sum_{n_d=0}^{\infty} P(N_d(t) = n_d) P(T_{n_d} = x_d) \right) \\
&\quad = \left(\sum_{n_1=0}^{\infty} P(N(t/d) = n_1) P(T_{n_1} = x_1) \right) \cdots \left(\sum_{n_d=0}^{\infty} P(N(t/d) = n_d) P(T_{n_d} = x_d) \right) \\
&\quad = P(Y_{t/d} = x_1) \cdots P(Y_{t/d} = x_d),
\end{aligned}$$

where $\{Y_t\}$ is a continuous time simple random walk on $\mathbf{Z}$. Summing over all possible values of $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d$ gives $P(R_t^{(j)} = x_j) = P(Y_{t/d} = x_j)$. Thus the $R_t^{(j)}$ are iid and have the same law as $Y_{t/d}$. It follows that

$$P(R_t = 0) = P(R_t^{(1)} = 0, \ldots, R_t^{(d)} = 0) = P(Y_{t/d} = 0) \cdots P(Y_{t/d} = 0) = P(Y_{t/d} = 0)^d.$$

(d) Let $\{S_n\}$ and $\{R_t\}$ be discrete and continuous time simple random walks on $\mathbf{Z}^d$. Assume that $\frac{c}{\sqrt{t}} \leq P(Y_t = 0) \leq \frac{C}{\sqrt{t}}$ for all $t \geq T$. Since $0 \leq \int_0^{Td} P(R_t = 0)\, dt \leq Td$, by part (a), $\{S_n\}$ is recurrent if and only if $\int_{Td}^{\infty} P(R_t = 0)\, dt < \infty$. Part (c) shows that $P(R_t = 0) = P(Y_{t/d} = 0)^d$. We compute

$$\begin{aligned}
\int_{Td}^{\infty} P(Y_{t/d} = 0)^d \, dt &\leq \int_{Td}^{\infty} \left(\frac{C}{\sqrt{t/d}} \right)^d dt = C^d d^{d/2} \int_{Td}^{\infty} \frac{1}{t^{d/2}} \, dt, \\
\int_{Td}^{\infty} P(Y_{t/d} = 0)^d \, dt &\geq \int_{Td}^{\infty} \left(\frac{c}{\sqrt{t/d}} \right)^d dt = c^d d^{d/2} \int_{Td}^{\infty} \frac{1}{t^{d/2}} \, dt.
\end{aligned}$$

When $d = 1$ or $d = 2$, $\int_{Td}^{\infty} (1/t^{d/2})\, dt$ diverges and so $\{S_n\}$ is recurrent. When $d \geq 3$, the integral converges and so $\{S_n\}$ is transient.