Math 6710, Fall 2016
Final Exam

Instructions: This is an open-book exam. You may consult any printed or online source, but you must explicitly cite all sources besides the textbook and lecture notes. Apart from asking me to clarify the questions, you may not get help from any person. In particular, you are not allowed to work with each other. The exam should be submitted via email or to my office, slid under the door if I am not there.

The exam is due on **Thursday, December 8** by **11:30 AM**.

1. Let $X_1, X_2, \ldots$ be iid random variables with mean zero and variance $0 < \sigma^2 < \infty$. Show that

$$\frac{X_1 + \cdots + X_n}{(X_1^2 + \cdots + X_n^2)^{1/2}} \Rightarrow Z$$

where $Z \sim N(0,1)$.

2. An Erdős-Rényi random graph on n vertices with parameter $0 \le p \le 1$ is constructed by taking the vertex set $V = \{v_1, \ldots, v_n\}$ and drawing an edge between each pair of distinct vertices v_i, v_j with independent probability p. Let E be the (random) set of edges; the expected number of edges is $p\binom{n}{2}$.

For each n, let (V_n, E_n) be an Erdős-Rényi random graph on n vertices with $p = 1/2$. (The graphs for different n are not necessarily independent.)

(a) Show that the number of edges $|E_n|$ satisfies a strong law of large numbers and a central limit theorem.

(b) Let T_n be the number of triangles in (V_n, E_n), that is, the number of triples $\{v_i, v_j, v_k\}$ such that $\{v_i, v_j\}, \{v_i, v_k\}, \{v_j, v_k\}$ are all in E_n. Show that $T_n / \binom{n}{3} \to 1/8$ almost surely. *Hint:* Beware the lack of independence. Look at the proof of the strong law that assumed finite fourth moments; for this problem you only need $E[T_n^2]$.

Note: There is also a central limit theorem for T_n, but proving it requires techniques that do a better job than the ones we developed at handling dependence.

3. (a) Suppose the random variables X_n converge to X a.s. and that $E[|X_n|] \to E[|X|] < \infty$. Prove that $E[|X_n - X|] \to 0$. *Hint:* Write each $X_n = X_{n,1} + X_{n,2}$, where $X_{n,1}$ and $X_{n,2}$ have the same sign and $X_{n,1}$ is dominated by $|X|$.

(b) State and prove the analogous statement for L^p, $p > 1$.

4. Let $X_1, X_2, \ldots$ be iid with distribution function F. The empirical distribution

function of $X_1, \ldots, X_n$ is $F_n(x) = \frac{1}{n} \cdot \#\{1 \le i \le n : X_i \le x\}$. The *Kolmogorov-Smirnov statistic* is

$$D_n^{(F)} = \sup_{x \in \mathbf{R}} |F_n(x) - F(x)|.$$

The Glivenko-Cantelli theorem says that for any distribution F, $D_n^{(F)} \to 0$ a.s.

(a) Let $U(x)$ be the distribution function of a Uniform$[0,1]$ random variable. Prove that for any continuous distribution function F, the random variables $D_n^{(F)}$ and $D_n^{(U)}$ are equal in distribution for each n. In other words, the law of the Kolmogorov-Smirnov statistic does not depend on F as long as F is continuous.

(b) Suppose now that F is not necessarily continuous. Prove that for all $y \ge 0$, $P(D_n^{(F)} > y) \le P(D_n^{(U)} > y)$. Thus, discontinuities in F can only make the Kolmogorov-Smirnov statistic "stochastically smaller."

5. Let $X_1, X_2, \ldots$ be iid $\mathbf{R}^d$-valued random variables and let $S_n = X_1 + \cdots + X_n$ ($S_0 = 0$) be the associated random walk. In this problem we define a continuous-time version of S_n. Let $N(t)$ be a rate 1 Poisson process independent of all the X_n, and set $R_t = S_{N(t)}$. Thus $\{R_t\}$ follows the same path as $\{S_n\}$ but takes steps at Poisson times instead of integer times.

(a) Assume that the X_n are $\mathbf{Z}^d$-valued. Show that $\{S_n\}$ is recurrent if and only if

$$\int_0^\infty P(R_t = 0)dt = \infty.$$

(b) In the general case when the X_n are $\mathbf{R}^d$-valued, find and verify a similar necessary and sufficient condition as in part (a) for recurrence of $\{S_n\}$.

(c) Suppose that S_n is the simple random walk on $\mathbf{Z}^d$ and let the continuous time version be $R_t = (R_t^{(1)}, \ldots, R_t^{(d)})$, where the $R_t^{(j)}$ are the individual coordinates. For each $t \ge 0$, prove that the $R_t^{(j)}$ are iid and that each one has the same law as $Y_{t/d}$, where Y_t is the continuous time simple random walk on $\mathbf{Z}$. Conclude that $P(R_t = 0) = P(Y_{t/d} = 0)^d$.

(interlude) To compute $P(Y_t = 0)$ heuristically, note that the number of steps taken by time t is Poisson(t), which is roughly $t + O(\sqrt{t})$. Half of the time, the number of steps is odd and so Y_t cannot be zero. The other half of the time, the number of steps is $2k$ where $k \approx t/2$ and then the probability that the simple random walk is at zero is $\frac{1}{2^{2k}}\binom{2k}{k} \approx \frac{1}{\sqrt{\pi k}} \approx \frac{\sqrt{2}}{\sqrt{\pi t}}$. This suggests that $P(Y_t = 0) \sim \frac{1}{\sqrt{2\pi t}}$.

(d) Assume (you do not have to prove) a weaker version of the statement above:

there exist constants $0 < c < C$ such that $\frac{c}{\sqrt{t}} \leq P(Y_t = 0) \leq \frac{C}{\sqrt{t}}$ for sufficiently large t. Use this along with parts (a) and (c) to show that simple random walk on $\mathbf{Z}^d$ is recurrent in dimensions $1, 2$ and transient in dimensions 3 and higher.