Grading: 1 (10 pts), 2a, 2b, 3a, 3b, 4a, 4b (each 5 pts).

Problem 1: Suppose otherwise, that there exists a countably infinite σ algebra $\mathcal{F}_{U_0}$ on a set U_0. Note that for each set $U \in \mathcal{F}$ s.t. $U \neq \{\emptyset, U_0\}$, $\mathcal{F}_U := \{U \cap V : V \in \mathcal{F}\}$ is a σ algebra on U_0. So, $\mathcal{F}_U$ and $\mathcal{F}_{U^c}$ are both σ algebras and since for any $V \in \mathcal{F}$, $V = (V \cap U) \cup (V \cap U^c)$, then one of $\mathcal{F}_U$, $\mathcal{F}_{U^c}$ is infinite, which we denote by $\mathcal{F}_{U_1}$. Thus, $U_1 \subset U_0$ and $\mathcal{F}_{U_1} \subset \mathcal{F}_{U_0}$ is infinite. Repeating the argument we hence get a sequence $U_1 \supset U_2 \supset ...$ of sets and $\mathcal{F}_{U_1} \supset \mathcal{F}_{U_2} \supset ...$ of infinite sigma algebras. As a result, the sets $V_i := U_i \setminus U_{i+1}$, $i = 1, 2, ...$ are disjoint and not equal to $\{\emptyset, U_i\}$. Next, we define a function $f : \mathcal{P}(\mathbb{N}) \to \mathcal{F}_{U_0}$ by $f(\mathcal{N}) = \cup_{i \in \mathcal{N}} V_i$ (countable union), which is a bijection since V_i's are disjoint and nonempty. Hence, the cardinality of $\mathcal{F}_{U_0}$ is same as the cardinality of $\mathcal{P}(\mathbb{N})$ i.e. uncountable, which contradicts the initial assumption.

Problem 2:

(a) First, lets recall the two definitions of the λ system. *Def. 1*: **a)** $\Omega \in \mathcal{L}$. **b)** if $A, B \in \mathcal{L}$ and $A \subseteq B$, then $B \setminus A \in \mathcal{L}$. **c)** if $A_1, A_2, ... \in \mathcal{L}$ and $A_n \subseteq A_{n+1}$ for all $n \geq 1$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{L}$. *Def. 2*: **a)** $\Omega \in \mathcal{L}$. **b)** if $A \in \mathcal{L}$ then $A^c \in \mathcal{L}$. **c)** if $A_1, A_2, ... \in \mathcal{L}$ and A_n are mutually disjoint for all $n \geq 1$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{L}$. We show that *Def. 1* implies *Def. 2*: 1a) and 1b) for $B = \Omega$ implies 2b). To show 2c), denote $B_i = \cup_{n=1}^{i} A_n$. Using 2b), if A and B are disjoint then $A \subseteq B^c$ and hence $(B^c \setminus A)^c = B \cup A \in \mathcal{L}$. Next, by induction, we have 2c) but for finite unions, which implies $B_i \in \mathcal{L}$ but then we can use 1c) to get 2c). The prove of the opposite direction is analogous (denote $B_i = A_i \setminus A_{i-1}$).

(b) Counterexample: take $\Omega = \{1, 2, 3, 4\}$, $\mathcal{L} = \{\emptyset, \{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \Omega\}$, then it is λ-system but not a σ algebra since not closed under intersections (e.g. $\{3, 4\} \cap \{2, 4\} = \{4\} \notin \mathcal{L}$).

Problem 3:

(a) The indicator function $\mathbf{1}_{E_n}(\omega)$ equals either 0 or 1, so $\limsup_{n\to\infty} \mathbf{1}_{E_n}(\omega) = 1$ iff the sequence $\mathbf{1}_{E_n}(\omega)$ takes the value 1 infinitely often, i.e. $\exists\ N \in \mathbb{N}$ s.t. $\forall\ m \geq N$: $\omega \in E_m$, that is $\omega \in \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} E_m$. The latter, by the definition is equivalent to $\mathbf{1}_{\limsup_{n\to\infty} E_n(\omega)} = 1$. The argument for $\mathbf{1}_{E_n}(\omega) = 0$ is analogous and since choice of ω was arbitrary, the proof is complete.

(b) Since $\cup_{m=n}^{\infty} E_m$ is a decreasing sequence, using the continuity from above property of the measure we get that $0 \leq P(E_n\ i.o.) = P(\cap_{n=1}^{\infty} \cup_{m=n}^{\infty} E_m) = \lim_{n\to\infty} P(\cup_{m=n}^{\infty} E_m)$. By combining the latter with the fact that $\sum_{n=1}^{\infty} P(E_n)$ implies $\lim_{n\to\infty} \sum_{m=n}^{\infty} P(E_m) = 0$, we get the desired result.

Problem 4: We construct an example which shows that $\mathcal{A}$ is not closed under intersections, which implies it is neither a σ algebra nor an algebra. Take the set $A = \{2, 4, 6, 8, 10, ...\}$ of even integers, then $A \in \mathcal{A}$. Next, we construct a set B in the following way: we begin with $\{2,3\}$ and starting with $k=2$, take all even numbers $2^k < n \leq (3/2) * 2^k$, and all odd numbers $(3/2) * 2^k < n \leq 2^{k+1}$. As a result, we can see that $A \cap B \notin \mathcal{A}$, because:

1. B contains exactly one of every consecutive pair $\{2m-1, 2m\}$ so it is easy to show that its asymptotic density is 1/2.

2. Starting with $k=2$, the number of elements of $A \cap B$ that are at most $(3/2) * 2^k$ is exactly 2^{k-1}, giving a density of 1/3.

3. Starting with $k=2$, the number of elements of $A \cap B$ that are at most 2^k is exactly 2^{k-2}, giving a density of 1/4.

The motivation was to start with A and design B to satisfy observation 1. Even elements of B increase the density of $A \cap B$, while odd elements of B decrease the density of $A \cap B$. Choose even elements until the density of $A \cap B$ reaches 1/3, then choose odd elements until the density of $A \cap B$ drops to 1/4, then choose more even elements until the density rises again to 1/3, etc. The resulting set B is the one above.