Math 6710, Fall 2016
Homework 1

1. Prove that there are no countably infinite σ-algebras.

2. Recall that a λ-system on Ω is a collection of subsets of Ω that includes Ω itself and is closed under subset differences and countable increasing unions.

(a) Show that a λ-system may be equivalently defined as a nonempty collection of subsets that is closed under complements and countable disjoint unions.

(b) Give an example of a λ-system that is not a σ-algebra.

3. The limit superior of a sequence of events $E_1, E_2, \ldots \in \Omega$ is defined as

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m.$$

Observe that $\omega \in \Omega$ is contained in $\limsup_{n\to\infty} E_n$ if and only if ω belongs to infinitely many of the events E_i. For this reason, we often write

$$\limsup_{n\to\infty} E_n = \{E_n \text{ i.o.}\}$$

where "i.o." stands for "infinitely often."

(a) Justify the nomenclature by showing that

$$\limsup_{n\to\infty} \mathbf{1}_{E_n} = \mathbf{1}_{\limsup_{n\to\infty} E_n}.$$

(Recall that the indicator function $\mathbf{1}_A$ is defined by

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

The limsup on the left is the usual one from analysis and the limsup on the right is the set theoretic one defined above.)

(b) Using the fact that $F_n = \bigcup_{m=n}^{\infty} E_m$ defines a countable decreasing sequence (i.e. each $F_n \supseteq F_{n+1}$), prove the first Borel-Cantelli Lemma:

$$\sum_{n=1}^{\infty} P(E_n) < \infty \quad \text{implies} \quad P(E_n \text{ i.o.}) = 0.$$

4. Textbook exercise 1.1.6. (An algebra on Ω is a nonempty collection of subsets of Ω that is closed under complements and finite unions.)