Grading: 1, 2, 3, 4 (each 10 pts)

Problem 1: Suppose X_n has density function F_n and X has density function F. Since $Y_n \Rightarrow c$, by the first proposition in the first supplement of the weak convergence lecture, we have $Y_n \to_p c$ also. For all $\epsilon > 0$,

$$P(X_n + Y_n \le x) = P(X_n + Y_n \le x \text{ and } |Y_n - c| > \epsilon) + P(X_n + Y_n \le x \text{ and } |Y_n - c| \le \epsilon)$$

$$\le P(|Y_n - c| > \epsilon) + P(X_n \le x - c + \epsilon).$$

We have $P(|Y_n - c| > \epsilon) \to 0$ since $Y_n \to_p c$. Assuming F is continuous at $x - c + \epsilon$,

$$P(X_n \le x - c + \epsilon) = F_n(x - c + \epsilon) \to F(x - c + \epsilon)$$

and also then we have

$$\limsup_{n\to\infty} P(X_n + Y_n \le x) \le F(x - c + \epsilon).$$

Sending $\epsilon \to 0$ along a sequence ϵ_n such that F is continuous at each $x - c + \epsilon_n$ gives

$$\limsup_{n\to\infty} P(X_n + Y_n \le x) \le F((x - c)^+) = F(x - c).$$

For the lower bound note that

$$P(X_n \le x - c - \epsilon) = P(X_n \le x - c - \epsilon \text{ and } |Y_n - c| > \epsilon) + P(X_n \le x - c - \epsilon \text{ and } |Y_n - c| \le \epsilon)$$

$$\le P(|Y_n - c| > \epsilon) + P(X_n + Y_n \le x).$$

As before, if F is continuous at $x - c - \epsilon$ then

$$F(x - c - \epsilon) \le \liminf_{n\to\infty} P(X_n + Y_n \le x).$$

Sending $\epsilon \to 0$ along a sequence ϵ_n such that F is continuous at each $x - c - \epsilon_n$ gives

$$\liminf_{n\to\infty} P(X_n + Y_n \le x) \ge F((x - c)^-)$$

which is equal to $F(x - c)$ if F is continuous at $x - c$. Combining our two results we obtain

$$\lim_{n\to\infty} P(X_n + Y_n \le x) = F(x - c) = P(X + c \le x),$$

so that $X_n + Y_n \Rightarrow X + c$. The "consequence" is trivially true by setting $Y_n = Z_n - X_n$ and

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$c = 0$. For the example, there are many choices but we give one where $X_n + Y_n$ does not converge weakly to any distribution. Define Z by $P(Z = 1) = P(Z = -1) = \frac{1}{2}$. Set each X_n to have the same law as Z. Define $Y_n = X_n$ for n even and $Y_n = -X_n$ for n odd. Each Y_n also has the same law as Z, so $X_n \Rightarrow Z$ and $Y_n \Rightarrow Z$. When n is even, $X_n + Y_n$ takes the values ± 2 each with probability $\frac{1}{2}$, and when n is odd, $X_n + Y_n = 0$ a.s. Therefore the sequence $\{X_n + Y_n\}$ has no weak limit.

Problem 2: For any real x and any $0 < \epsilon < 1$ we have: (1) If $X_n > x/(1-\epsilon)$ and $Y_n \geq c(1-\epsilon)$ then $X_nY_n > cx$. (2) If $X_nY_n > cx$ and $0 < Y_n \leq c(1+\epsilon)$ then $X_n > x/(1+\epsilon)$. It follows that

$$\begin{aligned} P(X_n \leq x/(1+\epsilon)) - P(Y_n > c(1+\epsilon)) - P(Y_n \leq 0) &\leq P(X_nY_n \leq cx) \\ &\leq P(X_n \leq x/(1-\epsilon)) + P(Y_n < c(1-\epsilon)). \end{aligned}$$

Since $Y_n \Rightarrow c$, it follows that $Y_n \to c$ in probability and so

$$\begin{aligned} \liminf_{n\to\infty} P(X_n \leq x/(1+\epsilon)) &\leq \liminf_{n\to\infty} P(X_nY_n \leq cx) \\ &\leq \limsup_{n\to\infty} P(X_nY_n \leq cx) \leq \limsup_{n\to\infty} P(X_n \leq x/(1-\epsilon)). \end{aligned}$$

Assume $x \neq 0$. By sending $\epsilon \to 0$ along a sequence $\{\epsilon_n\}$ such that the distribution function of X is continuous at each $x/(1+\epsilon_n)$ and $x/(1-\epsilon_n)$,

$$P(X < x) \leq \liminf_{n\to\infty} P(X_nY_n \leq cx) \leq \limsup_{n\to\infty} P(X_nY_n \leq cx) \leq P(X \leq x).$$

Thus $P(X_nY_n \leq cx) \to P(X \leq x)$ at all nonzero x for which $P(X = x) = 0$. This shows that $X_nY_n \Rightarrow cX$.

Problem 3: Let each X_n have distribution function F_n and let F be the right-continuous pointwise limit of the F_n (that is, $F_n(x) \to F(x)$ for all x at which F is continuous). The sub-probability measure ν is defined by $\nu((-\infty, x]) = F(x) - \lim_{y\to-\infty} F(y)$. Fix $g \in C_K(\mathbb{R})$ and choose $M > 0$ large enough that $g(x) = 0$ for all $|x| \geq M$, and such that F is continuous at M and $-M$. Define

$$Y_n = \begin{cases} -M & \text{if } X_n \leq -M, \\ X_n & \text{if } -M < X_n < M, \\ M & \text{if } X_n \geq M, \end{cases}$$

so that $E[g(Y_n)] = E[g(X_n)]$. If F_n' are the distribution functions of the Y_n, then

$$F_n'(x) = \begin{cases} 0 & \text{if } x < -M, \\ F_n(x) & \text{if } -M \le x < M, \\ 1 & \text{if } x \ge M \end{cases} \longrightarrow \begin{cases} 0 & \text{if } x < -M, \\ F(x) & \text{if } -M \le x < M, \\ 1 & \text{if } x \ge M \end{cases}$$

except possibly at discontinuity points of F in $(-M, M)$. It follows that $Y_n \Rightarrow Y$ where Y has the limiting distribution function above. Since g is bounded and continuous, as $n \to \infty$, $E[g(X_n)] = E[g(Y_n)]$ converges to

$$E[g(Y)] = g(-M)F(-M) + g(M)[1 - F(M)] + \int_{\mathbb{R}} g(x)\mathbf{1}\{|x| < M\} d\nu(x) = \int_{\mathbb{R}} g \, d\nu.$$

Now suppose $E[g(X_n)] \to \int_{\mathbb{R}} g \, d\nu$ for every real-valued continuous function with compact support. For each $a < b \in \mathbb{R}$ and $\epsilon > 0$, consider

$$g_{a,b,\epsilon}(x) = \begin{cases} 0, & x \le a - \epsilon \\ 1 - (a - x)/\epsilon, & a - \epsilon < x \le a \\ 1, & a < x \le b \\ 1 - (x - b)/\epsilon, & b < x \le b + \epsilon \\ 0, & x > b + \epsilon. \end{cases}$$

Each $g_{a,b,\epsilon} \in C_K(\mathbb{R})$. Hence, because $1_{[a,b]} \le g_{a,b,\epsilon} \le 1_{[a-\epsilon,b+\epsilon]}$ pointwise, we have

$$\begin{aligned} \limsup_{n\to\infty} P(a \le X_n \le b) = \limsup_{n\to\infty} E\left[1_{[a,b]} X_n\right] &\le \limsup_{n\to\infty} E[g_{a,b,\epsilon}(X_n)] \\ &= \int_{\mathbb{R}} g_{a,b,\epsilon} \, d\nu \le \int_{\mathbb{R}} 1_{[a-\epsilon,b+\epsilon]} \, d\nu = \nu([a - \epsilon, b + \epsilon]). \end{aligned}$$

Similarly, as long as ϵ is small enough that $a + \epsilon < b - \epsilon$,

$$\begin{aligned} \liminf_{n\to\infty} P(a \le X_n \le b) = \liminf_{n\to\infty} E\left[1_{[a,b]} X_n\right] &\ge \liminf_{n\to\infty} E\left[g_{a+\epsilon,b-\epsilon,\epsilon}(X_n)\right] \\ &= \int_{\mathbb{R}} g_{a+\epsilon,b-\epsilon,\epsilon} \, d\nu \ge \int_{\mathbb{R}} 1_{[a+\epsilon,b-\epsilon]} \, d\nu = \nu([a + \epsilon, b - \epsilon]). \end{aligned}$$

Sending $\epsilon \to 0$,

$$\nu((a, b)) \le \liminf_{n\to\infty} P(a \le X_n \le b) \le \limsup_{n\to\infty} P(a \le X_n \le b) \le \nu([a, b])$$

which implies that $P(a \le X_n \le b) \to \nu([a,b])$ as long as $\nu(\{a,b\}) = 0$.

The way we have defined things, it is not true that the random variables X_n must converge vaguely to any limit! Consider the following example: $P(X_{2n} = 0) = P(X_{2n} = 2n) = \frac{1}{2}$, $P(X_{2n+1} = 0) = P(X_{2n+1} = -(2n+1)) = \frac{1}{2}$. For any $g \in C_K(\mathbb{R})$, if n is large enough we have $E[g(X_n)] = \frac{1}{2}g(0) = \int_{\mathbb{R}} g\, d\nu$ where ν is a point mass at zero of weight $\frac{1}{2}$. However, the distribution functions F_n of X_n have no pointwise limit.

Problem 4: First, $e^{-xy}\sin(x)$ is integrable in the strip $0 < x < a$, $y > 0$ since

$$\int_0^a \int_0^\infty |e^{-xy}\sin(x)|\, dy\, dx \le \int_0^a \int_0^\infty e^{-xy} x\, dy\, dx = \int_0^a 1\, dx = a < \infty.$$

Hence, Fubini's theorem implies that

$$\int_0^a \int_0^\infty e^{-xy}\sin(x)\, dy\, dx = \int_0^\infty \int_0^a e^{-xy}\sin(x)\, dx\, dy. \tag{1}$$

The left-hand side of (1) is simplified as

$$\int_0^a \int_0^\infty e^{-xy}\sin(x)\, dy\, dx = \int_0^a \left(\frac{e^{-xy}\sin(x)}{-x}\Big|_{y=0}^{\infty} \right) dx = \int_0^a \frac{\sin(x)}{x}\, dx. \tag{2}$$

To compute the right-hand side of (1), use integration by parts:

$$I := \int_0^a e^{-xy}\sin(x)\, dx = 1 - \cos(a)e^{-ay} - y\int_0^a e^{-xy}\cos(x)\, dx = 1 - \cos(a)e^{-ay} - \sin(a)ye^{-ay} - y^2 I.$$

Therefore, the right-hand side of (1) is now equal to

$$\int_0^\infty \int_0^a e^{-xy}\sin(x)\, dx\, dy = \int_0^\infty I\, dy = \int_0^\infty \frac{1 - \cos(a)e^{-ay} - \sin(a)ye^{-ay}}{y^2+1}\, dy. \tag{3}$$

From (2) and (3), using that $\int_0^\infty \frac{1}{y^2+1}\, dy = \arctan(y)|_{y=0}^\infty = \frac{\pi}{2}$,

$$\int_0^a \frac{\sin(x)}{x}\, dx = \frac{\pi}{2} - \cos(a)\int_0^\infty \frac{e^{-ay}}{y^2+1}\, dy - \sin(a)\int_0^\infty \frac{ye^{-ay}}{y^2+1}\, dy.$$

Since $y^2 + 1 \ge 1$ and $y^2 + 1 \ge 2y \ge y$,

$$\left| \int_0^a \frac{\sin(x)}{x}\, dx - \frac{\pi}{2} \right| \le \left| \int_0^\infty \frac{e^{-ay}}{1}\, dy \right| + \left| \int_0^\infty \frac{ye^{-ay}}{y}\, dy \right| = \frac{2}{a}.$$

Sending a to ∞, we conclude that

$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}.$$