Math 6710, Fall 2016
Homework 10

1. Durrett Exercise 3.2.13. Provide an example where $X_n \Rightarrow X$ and $Y_n \Rightarrow Y$ but $X_n + Y_n \not\Rightarrow X + Y$. Of course X_n and Y_n must be defined on the same probability space for each n in order for this to make sense.

2. Durrett Exercise 3.2.14.

3. Prove that the sequence of random variables X_n converges vaguely to the sub-probability measure ν if and only if $E[g(X_n)] \to \int_{\mathbf{R}} g\, d\nu$ for all $g \in C_K(\mathbf{R})$, the space of real-valued continuous functions with compact support. *Hint:* For the forward direction, truncate outside the support of g. For the backward direction, consider quantities of the form $P(a \le X_n \le b)$.

4. Show that $\displaystyle\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ by solving Durrett Exercise 1.7.5.

Notes: (1) By "integrable" in the first sentence, it is meant that the integral of the absolute value is finite, which justifies the use of Fubini's Theorem.

(2) To integrate $e^{-xy} \sin x\, dx$, either integrate by parts twice or guess that the antiderivative will take the form $Ce^{-xy} \sin x + De^{-xy} \cos x$ and solve for C, D.

(3) The main formula in Exercise 1.7.5 is slightly incorrect! The correct version should allow you to express the difference

$$\int_0^a \frac{\sin x}{x} dx - \frac{\pi}{2}$$

as a sum of two integrals.

(4) The cryptic instruction to "replace $1 + y^2$ by 1" means to bound the first integral by noting that $1/(1 + y^2) \le 1/1$. For the second integral, use that $y/(1 + y^2) \le 1$ (it is easy to show that $y \le 1 + y^2$ by completing the square).

(5) This result could also be proved using contour integration, Fourier transforms, differentiation under the integral sign (à la Feynman), or other methods.