Grading: 1, 2, 3, 4, 5 (each 8 pts)

Problem 1: Since $|e^{-ita}| = 1$, by Fubini's theorem,

$$I_T = \frac{1}{2T}\int_{-T}^{T} e^{-ita}\varphi(t)dt = \frac{1}{2T}\int_{-T}^{T} e^{-ita}\int_{\mathbb{R}} e^{itx}d\mu(x)dt = \frac{1}{2T}\int_{\mathbb{R}}\int_{-T}^{T} e^{it(x-a)}dtd\mu(x).$$

Since $e^{it(x-a)} = \cos(t(x-a)) + i\sin(t(x-a))$ and $\sin(t(x-a))$ is an odd function of t,

$$I_T = \int_{\mathbb{R}} \frac{1}{2T}\int_{-T}^{T}\cos(t(x-a))dtd\mu(x).$$

We have

$$\frac{1}{2T}\int_{-T}^{T}\cos(t(x-a))dt = \begin{cases} 1, & x = a \\ \dfrac{\sin(T(x-a))}{T(x-a)}, & x \neq a. \end{cases}$$

In particular,

$$\left|\frac{1}{2T}\int_{-T}^{T}\cos(t(x-a))dt\right| \leq 1.$$

By the dominated convergence theorem,

$$\lim_{T\to\infty} I_T = \int_{\mathbb{R}}\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos(t(x-a))dt\right)d\mu(x)$$

and, since $|\sin(T(x-a))| \leq 1$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos(t(x-a))dt = \begin{cases} 1, & x = a \\ 0, & x \neq a. \end{cases}$$

It follows that

$$\lim_{T\to\infty} I_T = \int_{\mathbb{R}} \mathbf{1}\{x = a\}d\mu(x) = \mu(\{a\}).$$

Problem 2: Since $X_n \Rightarrow X_\infty$ and $Y_n \Rightarrow Y_\infty$, we have $\varphi_{X_n}(t) \to \varphi_{X_\infty}(t)$ and $\varphi_{Y_n}(t) \to \varphi_{Y_\infty}(t)$ for each t. Because X_n and Y_n are independent,

$$\varphi_{X_n+Y_n}(t) = \varphi_{X_n}(t)\cdot\varphi_{Y_n}(t) \longrightarrow \varphi_{X_\infty}(t)\cdot\varphi_{Y_\infty}(t) = \varphi_{X_\infty+Y_\infty}(t).$$

Since $\varphi_{X_\infty}(t)$ and $\varphi_{Y_\infty}(t)$ are continuous at 0, so is $\varphi_{X_\infty+Y_\infty}(t)$. Hence, the continuity theorem implies that $X_n + Y_n \Rightarrow X_\infty + Y_\infty$.

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Problem 3: We proceed by induction. Base case $n = 0$: $\varphi^{(0)}(t) = \varphi(t) = \int_{\mathbb{R}} (ix)^0 e^{itx} \mu(dx)$, and it is continuous by Theorem 3.3.1 in Durrett (or see the proof of continuity below which also works for $n = 0$). Inductive step: Suppose $\int_{\mathbb{R}} |x|^n \mu(dx) < \infty$. Then the same is true for the $(n-1)$st power, so by the inductive hypothesis, $\varphi^{(n-1)}(t) = \int_{\mathbb{R}} (ix)^{n-1} e^{itx} \mu(dx)$. Thus

$$\varphi^{(n)}(t) = \lim_{h \to 0} \frac{\varphi^{(n-1)}(t+h) - \varphi^{(n-1)}(t)}{h} = \lim_{h \to 0} \int_{\mathbb{R}} (ix)^{n-1} e^{itx} \frac{e^{ihx} - 1}{h} \mu(dx).$$

To apply dominated convergence we must find an upper bound for

$$\left| (ix)^{n-1} e^{itx} \frac{e^{ihx} - 1}{h} \right| = |x|^n \left| \frac{e^{ihx} - 1}{hx} \right|$$

that does not depend on h. Let

$$g(y) = \frac{e^{iy} - 1}{y} = i + \frac{i^2 y}{2!} + \frac{i^3 y^2}{3!} + \cdots .$$

Since g is continuous (in particular, $g(0) = i$) we know that $\max\{|g(y)| : y \in [-1, 1]\} < \infty$. For $|y| \geq 1$ we use that the numerator $|e^{iy} - 1| \leq 2$, so $|g(y)| \leq 2$. Hence there is C such that $|g(y)| \leq C < \infty$ for all $y \in \mathbb{R}$, and we can take

$$\left| (ix)^{n-1} e^{itx} \frac{e^{ihx} - 1}{h} \right| \leq C|x|^n$$

which is integrable with respect to $\mu(dx)$ by assumption. Dominated convergence now gives

$$\varphi^{(n)}(t) = \int_{\mathbb{R}} (ix)^{n-1} e^{itx} (ix) \mu(dx) = \int_{\mathbb{R}} (ix)^n e^{itx} \mu(dx).$$

To prove continuity,

$$\lim_{h \to 0} \varphi^{(n)}(t+h) = \lim_{h \to 0} \int_{\mathbb{R}} (ix)^n e^{i(t+h)x} \mu(dx).$$

The integrand is dominated by $|x|^n$, which is integrable, so dominated convergence gives

$$\lim_{h \to 0} \varphi^{(n)}(t+h) = \int_{\mathbb{R}} (ix)^n e^{itx} \mu(dx) = \varphi^{(n)}(t).$$

Problem 4: Suppose $E[X_i^2] = \infty$. Let $X_1', X_2', \ldots$ be an independent copy of the original sequence, and $Y_i = X_i - X_i'$. Fix $K > 0$. Given $A > 0$ (whose value will be determined later), let $U_i = Y_i 1_{|Y_i| \leq A}$ and $V_i = Y_i 1_{|Y_i| > A}$. Since $Y_i = U_i + V_i$, if $\sum_{i=1}^{n} U_i \geq K\sqrt{n}$ and $\sum_{i=1}^{n} V_i \geq 0$, then $\sum_{i=1}^{n} Y_i \geq K\sqrt{n}$. Hence,

$$P\left(\sum_{i=1}^{n} Y_i \geq K\sqrt{n}\right) \geq P\left(\sum_{i=1}^{n} U_i \geq K\sqrt{n},\ \sum_{i=1}^{n} V_i \geq 0\right)$$
$$= P\left(\sum_{i=1}^{n} U_i \geq K\sqrt{n}\right) P\left(\sum_{i=1}^{n} V_i \geq 0 \;\middle|\; \sum_{i=1}^{n} U_i \geq K\sqrt{n}\right).$$

Because of the symmetry between X_i and X_i',

$$P\left(\sum_{i=1}^{n} V_i \geq 0 \;\middle|\; \sum_{i=1}^{n} U_i \geq K\sqrt{n}\right) = P\left(\sum_{i=1}^{n} V_i \leq 0 \;\middle|\; \sum_{i=1}^{n} U_i \geq K\sqrt{n}\right).$$

Since the sum of these conditional probabilities is at least 1, both of them have to be at least 1/2. Hence,

$$P\left(\sum_{i=1}^{n} U_i \geq K\sqrt{n},\ \sum_{i=1}^{n} V_i \geq 0\right) \geq \frac{1}{2} P\left(\sum_{i=1}^{n} U_i \geq K\sqrt{n}\right).$$

The central limit theorem implies that $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i \Rightarrow \sqrt{\mathrm{Var}(U_i)} Z$ where $Z \sim \mathcal{N}(0,1)$. Hence,

$$P\left(\sum_{i=1}^{n} U_i \geq K\sqrt{n}\right) \longrightarrow P\left(Z \geq \frac{K}{\sqrt{\mathrm{Var}(U_i)}}\right)$$

as $n \to \infty$. By monotone convergence, $\mathrm{Var}(U_i) = E[U_i^2] \nearrow E[Y_i^2] = \infty$ as $A \to \infty$, so by setting A sufficiently large (depending on K) we can ensure that $P(Z \geq K/\sqrt{\mathrm{Var}(U_i)}) \geq .45$ and therefore $P(\sum_{i=1}^{n} U_i \geq K\sqrt{n}) \geq 2/5$ for sufficiently large n. Combining all the inequalities, we have shown that for large enough n depending on K,

$$P\left(\sum_{i=1}^{n} Y_i \geq K\sqrt{n}\right) \geq \frac{1}{5}.$$

On the other hand, we assumed that $S_n/\sqrt{n}$ converges weakly to some limit W. By Problem 2, $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i \Rightarrow W - W'$ where W' is an independent copy of W, so

$$P\left(\sum_{i=1}^{n} Y_i \geq K\sqrt{n}\right) \longrightarrow P(W - W' \geq K)$$

(as long as $P(W - W' = K) = 0$ for continuity purposes). Setting K large enough that $P(W - W' \geq K) < 1/5$ gives a contradiction.

Problem 5: The characteristic function for X_k is

$$\varphi_{X_k}(t) = E[e^{itX_k}] = \frac{1}{2k}\int_{-k}^{k} e^{itx}dx = \frac{1}{2k}\int_{-k}^{k}\cos(tx)dx = \frac{1}{kt}\sin(kt) \approx 1 - \frac{1}{6}(kt)^2.$$

Consequently, the characteristic function for $n^{-\alpha}S_n$ is given by

$$\varphi_n(t) = \prod_{k=1}^{n}\varphi_{X_k}(n^{-\alpha}t) \approx \prod_{k=1}^{n}\left(1 - \frac{(kt)^2}{6n^{2\alpha}}\right).$$

Let

$$c_{k,n} = \varphi_{X_k}(n^{-\alpha}t) - 1 = \frac{\sin(kn^{-\alpha}t)}{kn^{-\alpha}t} - 1 \approx -\frac{(kt)^2}{6n^{2\alpha}}.$$

We aim to apply Fact 11.2 from the notes. First,

$$\max_{1\le k\le n}|c_{k,n}| \le \max\left\{\left|\frac{\sin(x)}{x} - 1\right| : |x| \le n^{1-\alpha}|t|\right\}$$

which, as long as $\alpha > 1$, converges to 0 as $n \to \infty$ because $\sin(x)/x \to 1$ as $x \to 0$. Indeed,

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{5!} - \cdots$$

which implies that there is a constant C such that

$$g(x) := \frac{\sin(x)}{x} - \left(1 - \frac{x^2}{6}\right)$$

satisfies $|g(x)| \le Cx^4$ for all $|x| \le 1$. Thus, if n is large enough that $n^{1-\alpha}|t| \le 1$,

$$\sum_{k=1}^{n} c_{k,n} = \sum_{k=1}^{n}\left(\frac{-(kt)^2}{6n^{2\alpha}} + g(kn^{-\alpha}t)\right) = \frac{-t^2n(n+1)(2n+1)}{36n^{2\alpha}} + \sum_{k=1}^{n} g(kn^{-\alpha}t)$$

and

$$\sum_{k=1}^{n}|g(kn^{-\alpha}t)| \le \sum_{k=1}^{n} Ck^4n^{-4\alpha}t^4 \le \sum_{k=1}^{n} Cn^{4-4\alpha}t^4 = Cn^{5-4\alpha}t^4.$$

If we choose $\alpha = 3/2$ then $\sum_{k=1}^{n}|g(kn^{-\alpha}t)| \to 0$ as $n \to \infty$ and so

$$\lim_{n\to\infty}\sum_{k=1}^{n} c_{k,n} = -\frac{t^2}{18}.$$

Finally,

$$\limsup_{n\to\infty}\sum_{k=1}^{n}|c_{k,n}| = \limsup_{n\to\infty}\sum_{k=1}^{n}\left|\frac{-(kt)^2}{6n^3}+g(kn^{-3/2}t)\right|$$
$$\le \limsup_{n\to\infty}\sum_{k=1}^{n}\frac{(kt)^2}{6n^3}+\limsup_{n\to\infty}\sum_{k=1}^{n}|g(kn^{-3/2}t)| = \frac{t^2}{18}+0<\infty.$$

Hence, Fact 11.2 implies that as $n\to\infty$,

$$\varphi_n(t)=\prod_{k=1}^{n}(1+c_{k,n})\longrightarrow e^{-t^2/18}=\varphi_Z(t) \text{ where } Z\sim\mathcal{N}\left(0,\frac{1}{9}\right).$$

We conclude that $\alpha=3/2$, $\mu=0$, and $\sigma^2=1/9$.

Note: Some students used the Lindeberg-Feller CLT to solve this problem. That was not the intended solution (because it was supposed to be solvable without having seen Lindeberg-Feller yet) but it also should work.