Grading: 1, 2, 3, 4, 5 (each 8 pts)

Problem 1: Since the distribution of X_m is symmetric around 0, $E[X_m] = 0$. Hence,

$$\mathrm{Var}(X_m) = \frac{1}{2m^2}\left(m^2 + (-m)^2\right) + \frac{1}{2}\left(1 - \frac{1}{m^2}\right)\left(1^2 + (-1)^2\right) = 2 - \frac{1}{m^2}.$$

Since the X_m are independent,

$$\frac{1}{n}\mathrm{Var}(S_n) = \frac{1}{n}\sum_{m=1}^{n}\mathrm{Var}(X_m) = \frac{1}{n}\sum_{m=1}^{n}\left(2 - \frac{1}{m^2}\right) = 2 - \frac{1}{n}\sum_{m=1}^{n}\frac{1}{m^2} \longrightarrow 2.$$

Let $Y_m = \mathrm{sgn}(X_m)$ ($= +1$ if $X_m > 0$ and -1 if $X_m < 0$) and $Z_m = X_m - Y_m$. Note that $P(Z_m \neq 0) = 1/m^2$. Since the Y_m are iid with mean 0 and variance 1, the CLT implies that $\sum_{m=1}^{n} Y_m/\sqrt{n} \Rightarrow \mathcal{N}(0,1)$. Also, since $\sum_{m=1}^{\infty} P(Z_m \neq 0) < \infty$, the first Borel-Cantelli Lemma implies that $P(Z_m \neq 0 \text{ i.o.}) = 0$, thus $Z_m \to 0$ a.s. and in probability. The Converging Together Lemma (Homework 10, Problem 1) now implies that

$$\frac{1}{\sqrt{n}}S_n = \frac{1}{\sqrt{n}}\sum_{m=1}^{n} Y_m + \frac{1}{\sqrt{n}}\sum_{m=1}^{n} Z_m \Rightarrow \mathcal{N}(0,1).$$

Problem 2: Let $\overline{X}_m = X_m - E[X_m]$ and $X_{n,m} = \overline{X}_m/\alpha_n$. Then $E[X_{n,m}] = 0$ and by definition,

$$\lim_{n\to\infty}\sum_{m=1}^{n} E[X_{n,m}^2] = \lim_{n\to\infty}\alpha_n^{-2}\sum_{m=1}^{n} E[\overline{X}_m^2] = \lim_{n\to\infty}\alpha_n^{-2}\,\mathrm{Var}(S_n) = 1.$$

To check the Lindeberg condition, for any fixed $\epsilon > 0$,

$$\begin{aligned}
\sum_{m=1}^{n} E[X_{n,m}^2; |X_{n,m}| > \epsilon] &= \alpha_n^{-2}\sum_{m=1}^{n} E[\overline{X}_m^2; |\overline{X}_m| > \epsilon\alpha_n] \\
&\leq \alpha_n^{-2}\sum_{m=1}^{n} E\left[\overline{X}_m^2\frac{|\overline{X}_m|^\delta}{(\epsilon\alpha_n)^\delta}; |\overline{X}_m| > \epsilon\alpha_n\right] \\
&\leq \epsilon^{-\delta}\alpha_n^{-(2+\delta)}\sum_{m=1}^{n} E[|\overline{X}_m|^{2+\delta}] \longrightarrow 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Consequently, the Lindeberg-Feller CLT implies that

$$\frac{1}{\alpha_n}(S_n - E[S_n]) = \sum_{m=1}^{n} X_{n,m} \Rightarrow Z \sim \mathcal{N}(0,1).$$

[1] Contributed by Binh Tang

Problem 3: Since $E[X_{n,m}] = p_{n,m}$, we have $E[\overline{X}_{n,m}^2] = E[(X_{n,m} - p_{n,m})^2] = p_{n,m}(1 - p_{n,m})$. Hence,

$$\lim_{n\to\infty} \sum_{m=1}^{n} E[\overline{X}_{n,m}^2] = \lim_{n\to\infty} \sum_{m=1}^{n} p_{n,m}(1 - p_{n,m}) = \lambda - \lim_{n\to\infty} \sum_{m=1}^{n} p_{n,m}^2 = \lambda$$

because it follows from $\max_{1\le m\le n} p_{n,m} \to 0$ that

$$0 \le \sum_{m=1}^{n} p_{n,m}^2 \le \left(\max_{1\le m\le n} p_{n,m}\right)\left(\sum_{m=1}^{n} p_{n,m}\right) \longrightarrow 0 \cdot \lambda = 0.$$

Since $E[\overline{X}_{n,m}] = 0$, if the Lindeberg condition were satisfied then

$$S_n - \sum_{m=1}^{n} p_{m,n} = \sum_{m=1}^{n} \overline{X}_{n,m} \Rightarrow \sqrt{\lambda} Z, \text{ where } Z \sim \mathcal{N}(0,1),$$

and so $S_n \Rightarrow \lambda + \sqrt{\lambda}Z$ by observing that $\sum_{m=1}^{n} p_{n,m} \to \lambda$ (and using the Converging Together Lemma, if you like). Direct computation of the density function shows that $\lambda + \sqrt{\lambda}Z \sim \mathcal{N}(\lambda, \lambda)$. However, we have proved that $S_n \Rightarrow \text{Poisson}(\lambda)$, and thus the Lindeberg condition cannot hold. This can be seen by direct computation. Fix $0 < \epsilon < 1/2$ and choose n large enough that each $p_{n,m} < \epsilon$ (and thus also $\epsilon < 1 - p_{n,m}$). We know that $\overline{X}_{n,m} = 1 - p_{n,m}$ with probability $p_{n,m}$ and $\overline{X}_{n,m} = -p_{n,m}$ with probability $1 - p_{n,m}$, so

$$\sum_{m=1}^{n} E[\overline{X}_{n,m}^2; |\overline{X}_{n,m}| > \epsilon] = \sum_{m=1}^{n} p_{n,m}(1 - p_{n,m})^2 = \sum_{m=1}^{n} (p_{n,m} - 2p_{n,m}^2 + p_{n,m}^3).$$

As $n \to \infty$, $\sum_{m=1}^{n} p_{n,m} \to \lambda$; $\sum_{m=1}^{n} p_{n,m}^2 \to 0$ (as shown above); and $\sum_{m=1}^{n} p_{n,m}^3 \le \sum_{m=1}^{n} p_{n,m}^2 \to 0$. Therefore,

$$\lim_{n\to\infty} \sum_{m=1}^{n} E[\overline{X}_{n,m}^2; |\overline{X}_{n,m}| > \epsilon] = \lambda > 0$$

and the Lindeberg condition is not satisfied.

Problem 4: Since $P(T > t + s \mid T > t) = P(T > s)$, we have $P(T > t + s) = P(T > t)P(t > s)$. Hence,

$$P(T > t) = P(T > t/2)^2 = \cdots = P(T > t/2^n)^{2^n}.$$

If there is $t > 0$ such that $P(T > t) = 0$, then $P(T > t/2^n) = 0$. Sending $n \to \infty$, we have $P(T > 0) = 0$, a contradiction. Therefore, $P(T > t) > 0$ for all t. Since $P(T > 1) > 0$, we can define $\lambda > 0$ with $e^{-\lambda} = P(T > 1)$. Thus, $P(T > 2^{-n})^{2^n} = e^{-\lambda}$, so $P(T > 2^{-n}) = e^{-\lambda 2^{-n}}$. Consequently, for every integer m,

$$P(T > m2^{-n}) = P(T > (m-1)2^{-n})P(T > 2^{-n}) = \cdots = P(T > 2^{-n})^m = e^{-\lambda m 2^{-n}}.$$

Given $t \geq 0$, $P(T > t) = \lim_{s \searrow t} P(T > s)$. For each n, let m_n be the least integer such that $s_n := m_n 2^{-n} \geq t$. Then $s_n \searrow t$ and

$$P(T > t) = \lim_{n\to\infty} P(T > s_n) = \lim_{n\to\infty} e^{-\lambda s_n} = e^{-\lambda t}.$$

Problem 5: N_j has a Poisson distribution with mean λp_j since

$$\begin{aligned} P(N_j = m_j) &= \sum_{m=m_j}^{\infty} P(N_j = m_j \mid N = m) P(N = m) \\ &= \sum_{m=m_j}^{\infty} \left(\binom{m}{m_j} p_j^{m_j} (1-p_j)^{m-m_j} \right) \left(e^{-\lambda} \frac{\lambda^m}{m!} \right) \\ &= \frac{(\lambda p_j)^{m_j}}{m_j!} e^{-\lambda} \sum_{m=m_j}^{\infty} \frac{(\lambda(1-p_j))^{m-m_j}}{(m-m_j)!} = \left(\frac{(\lambda p_j)^{m_j}}{m_j!} \right) e^{-\lambda} e^{\lambda(1-p_j)} = e^{-\lambda p_j} \frac{(\lambda p_j)^{m_j}}{m_j!}. \end{aligned}$$

In addition, $N_0, N_1, \ldots, N_k$ are independent because if $m = \sum_{j=1}^{k} m_j$ then

$$\begin{aligned} &P(N_0 = m_0, \ldots, N_k = m_k) = P(N_0 = m_0, \ldots, N_k = m_k \mid N = m) P(N = m) \\ &= \left(\frac{m!}{m_0! \ldots m_k!} p_0^{m_0} \ldots p_k^{m_k} \right) \left(e^{-\lambda} \frac{\lambda^m}{m!} \right) = \prod_{j=0}^{k} e^{-\lambda p_j} \frac{(\lambda p_j)^{m_j}}{m_j!} = P(N_0 = m_0) \cdots P(N_k = m_k). \end{aligned}$$