Grading: TBD

Problem 1: First, $\mathcal{P} \subseteq \mathcal{L}$. If $J \in \mathcal{P}$ then $J \in \sigma(X_1, \dots, X_k)$ for some k. Take $I_n = J$ for $n \geq k$; then each $I_n \in \sigma(X_1, \dots, X_n)$ and $P(I_n \Delta J) = 0$ for $n \geq k$, so $J \in \mathcal{L}$. To show that $\mathcal{L}$ is a λ-system, we must check that $\Omega \in \mathcal{L}$ (which is already implied by $\mathcal{P} \subseteq \mathcal{L}$) and that $\mathcal{L}$ is closed under subset differences and countable increasing unions.

Subset differences: Suppose $A, B \in \mathcal{L}$ with $A \subseteq B$. Find sequences $\{I_n\}, \{J_n\}$ such that each $I_n, J_n \in \sigma(X_1, \dots, X_n)$ and $P(I_n \Delta A), P(J_n \Delta B) \to 0$ as $n \to \infty$. To show that $B \setminus A \in \mathcal{L}$ it suffices to show that $P((J_n \setminus I_n)\Delta(B \setminus A)) \to 0$ as $n \to \infty$, since each $J_n \setminus I_n \in \sigma(X_1, \dots, X_n)$.

$$\begin{aligned}
(J_n \setminus I_n)\Delta(B \setminus A) &= [(J_n \cap I_n^c) \cap (B \cap A^c)^c] \cup [(J_n \cap I_n^c)^c \cap (B \cap A^c)] \\
&= [(J_n \cap I_n^c) \cap (B^c \cup A)] \cup [(J_n^c \cup I_n) \cap (B \cap A^c)] \\
&= (J_n \cap I_n^c \cap B^c) \cup (J_n \cap I_n^c \cap A) \cup (J_n^c \cap B \cap A^c) \cup (I_n \cap B \cap A^c) \\
&\subseteq (J_n \cap B^c) \cup (I_n^c \cap A) \cup (J_n^c \cap B) \cup (I_n \cap A^c) \\
&= (J_n \Delta B) \cup (I_n \Delta A).
\end{aligned}$$

Therefore, $P((J_n \setminus I_n)\Delta(B \setminus A)) \leq P(J_n \Delta B) + P(I_n \Delta A) \to 0$ as $n \to \infty$.

Countable increasing unions: Suppose $A_m \nearrow A$ and each $A_m \in \mathcal{L}$. For each m, find sets $I_{m,n} \in \sigma(X_1, \dots, X_n)$ such that $P(I_{m,n} \Delta A_m) \to 0$ as $n \to \infty$. Let $n(m)$ be the least integer greater than or equal to m such that $P(I_{m,n(m)} \Delta A_m) \leq \frac{1}{m}$. In particular, $n(1) = 1$.

Since $P(I_{m,n(m)} \Delta A_m) \to 0$ and $P(A \setminus A_m) \to 0$ as $m \to \infty$, it is not hard to show that $P(I_{m,n(m)} \Delta A) \to 0$ as $m \to \infty$. This would suffice except that $I_{m,n(m)}$ is contained in $\sigma(X_1, \dots, X_{n(m)})$ rather than $\sigma(X_1, \dots, X_m)$. The solution is to slow down the sequence $\{I_{m,n(m)}\}$ by repeating terms. For example, if the sequence begins $(I_{1,1}, I_{2,4}, I_{3,6}, \dots)$ we would define $(J_1, J_2, \dots) = (I_{1,1}, I_{1,1}, I_{1,1}, I_{2,4}, I_{2,4}, I_{3,6}, \dots)$ so that each $J_k \in \sigma(X_1, \dots, X_k)$. The following argument makes this rigorous.

For each $k \geq 1$, let $m(k)$ be the greatest integer such that $n(m(k)) \leq k$. Since $n(1) \leq k$ and $n(m) \geq m > k$ for all $m > k$, $m(k)$ is well-defined and at most k. Set $J_k = I_{m(k),n(m(k))} \in \sigma(X_1, \dots, X_k)$. We show that $P(J_k \Delta A) \to 0$ as $k \to \infty$. First note that for all $M \geq 1$, if $k \geq n(M)$ then $m(k) \geq M$. Thus $m(k) \to \infty$ as $k \to \infty$. We have

$$\begin{aligned}
P(J_k \Delta A) &= P(J_k \setminus A) + P(A_{m(k)} \setminus J_k) + P((A \setminus A_{m(k)}) \setminus J_k) \\
&\leq P(J_k \setminus A_{m(k)}) + P(A_{m(k)} \setminus J_k) + P(A \setminus A_{m(k)}) \\
&= P(J_k \Delta A_{m(k)}) + P(A \setminus A_{m(k)}).
\end{aligned}$$

By construction, $P(J_k \Delta A_{m(k)}) = P(I_{m(k),n(m(k))} \Delta A_{m(k)}) \leq \frac{1}{m(k)} \to 0$ as $k \to \infty$. Also, since $A_m \nearrow A$, $P(A \setminus A_{m(k)}) \to 0$ as $k \to \infty$. Hence $P(J_k \Delta A) \to 0$ as $k \to \infty$, so $A \in \mathcal{L}$.

Problem 2: Since $\{Y_n\}$ is a Cauchy sequence in the complete metric space L^2, it has a limit which we call Z. So $Y_n \to Z$ in L^2 and therefore in probability, and there is a subsequence Y_{n_m} that converges to Z almost surely. However, we know that Y_n (and therefore also Y_{n_m}) converges to Y almost surely. Hence $Y = Z$ except on a set of measure zero and $Y_n \to Y$ in L^2. By the triangle inequality in L^2,

$$\|Y\|_2 - \|Y_n\|_2 \leq \|Y - Y_n\|_2, \quad \|Y_n\|_2 - \|Y\|_2 \leq \|Y_n - Y\|_2$$

which means $\Big| \|Y_n\|_2 - \|Y\|_2 \Big| \leq \|Y_n - Y\|_2 \to 0$ as $n \to \infty$ and so $\|Y_n\|_2 \to \|Y\|_2$. It follows that $E[Y_n^2] = \|Y_n\|_2^2 \to \|Y\|_2^2 = E[Y^2]$.

Problem 3: Case (i) is only possible when $X_1 = 0$ almost surely, which contradicts $E[X_1^2] > 0$. To rule out case (ii), assume for contradiction that $S_n \to \infty$ a.s. Let $T = \max\{n \geq 0 : S_n \leq 0\}$, so that $T < \infty$ a.s., and choose N such that $P(T > N) \leq \frac{1}{4}$. By the CLT, $S_n/(\sigma\sqrt{n}) \Rightarrow Z \sim N(0,1)$ and so

$$P(S_n \leq 0) = P\left(\frac{S_n}{\sigma\sqrt{n}} \leq 0\right) \to P(Z \leq 0) = \frac{1}{2} \quad \text{as } n \to \infty.$$

Choose $n > N$ large enough that $P(S_n \leq 0) \geq \frac{1}{3}$. However, if $S_n \leq 0$ then $T \geq n > N$, so also $P(T > N) \geq \frac{1}{3}$, which is a contradiction. This rules out case (ii), and case (iii) is also impossible by symmetry, so case (iv) is all that remains.

Problem 4: **(a)** By Proposition 17.1 in the notes, $\{S \leq n\}, \{T \leq n\} \in \sigma(X_1, \ldots, X_n)$ for each n. Therefore,

$$\{S \wedge T \leq n\} = \{S \leq n\} \cup \{T \leq n\} \in \sigma(X_1, \ldots, X_n),$$
$$\{S \vee T \leq n\} = \{S \leq n\} \cap \{T \leq n\} \in \sigma(X_1, \ldots, X_n)$$

and so $S \wedge T$, $S \vee T$ are stopping times, again by Proposition 17.1.

(b) Yes, $S + T$ is a stopping time: $\{S + T = n\} = \cup_{k=1}^n (\{S = k\} \cap \{T = n - k\})$, and each $\{S = k\} \cap \{T = n - k\} \in \sigma(X_1, \ldots, X_{k \vee (n-k)}) \subseteq \sigma(X_1, \ldots, X_n)$. No, $T - S$ is not necessarily a stopping time. For a counterexample, let the X_n be iid, $P(X_n = 0) = P(X_n = 1) = \frac{1}{2}$. Set $S = 10$ and $T = 11$ if $X_{11} = 0$, $T = 12$ if $X_{11} = 1$. Both S and T are stopping times, but $T - S$ is 1 or 2 depending on the value of X_{11}, so for example $\{T - S = 1\} \notin \sigma(X_1)$.

Problem 5: Use the notation of Example 4.1.4 in the textbook. First suppose $P(\alpha < \infty) < 1$. Set $K = \sup\{k \geq 0 : \alpha_k < \infty\}$, so that $P(K \geq k) = P(\alpha_k < \infty) = P(\alpha < \infty)^k \to 0$ as $k \to \infty$. Thus K is a.s. finite. When $K < \infty$ we have $S_n \leq S_{\alpha_K}$ for all $n \geq \alpha_K$, so $\limsup_{n\to\infty} S_n < \infty$ with probability 1.

Suppose conversely that $P(\alpha < \infty) = 1$, so that each α_k is a.s. finite. Theorem 4.1.4 in the textbook implies that the increments $Y_k := S_{\alpha_k} - S_{\alpha_{k-1}}$ are iid, and since each $Y_k > 0$ we have $0 < E[Y_k] \leq \infty$. In case $E[Y_k] = \mu < \infty$, the Strong Law of Large Numbers says that

$$\frac{S_{\alpha_k}}{k} = \frac{S_{\alpha_k} - S_{\alpha_0}}{k} = \frac{Y_1 + \cdots + Y_k}{k} \to \mu \quad \text{a.s.}$$

and so $S_{\alpha_k} \geq k\mu/2$ for sufficiently large k. In particular, $\limsup_{n\to\infty} S_n = \infty$ with probability 1. In case $E[Y_k] = \infty$, Theorem 9.2 in the notes implies that $S_{\alpha_k}/k \to \infty$ a.s. which again means that $\limsup_{n\to\infty} S_n = \infty$ with probability 1.

We use symmetric reasoning to show that if $P(\beta < \infty) < 1$ then $\liminf_{n\to\infty} S_n > -\infty$ with probability 1, and if $P(\beta < \infty) = 1$ then $\liminf_{n\to\infty} S_n = -\infty$ with probability 1. The four possibilities are:

$$P(\alpha < \infty) < 1 \text{ and } P(\beta < \infty) < 1 \implies -\infty < \liminf_{n\to\infty} S_n \leq \limsup_{n\to\infty} S_n < \infty \text{ a.s.}$$
$$P(\alpha < \infty) = 1 \text{ and } P(\beta < \infty) < 1 \implies -\infty < \liminf_{n\to\infty} S_n \leq \limsup_{n\to\infty} S_n = \infty \text{ a.s.}$$
$$P(\alpha < \infty) < 1 \text{ and } P(\beta < \infty) = 1 \implies -\infty = \liminf_{n\to\infty} S_n \leq \limsup_{n\to\infty} S_n < \infty \text{ a.s.}$$
$$P(\alpha < \infty) = 1 \text{ and } P(\beta < \infty) = 1 \implies -\infty = \liminf_{n\to\infty} S_n < \limsup_{n\to\infty} S_n = \infty \text{ a.s.}$$

If the first possibility holds, cases (ii),(iii),(iv) in Durrett Theorem 4.1.2 are ruled out, so case (i) must be true. Likewise, the second possibility rules out all but case (ii) and the third possibility rules out all but case (iii). The fourth possibility is identical to case (iv). Hence the four possibilities above correspond exactly to the four cases of the theorem.

Problem 6: **(i)** We have $E[X_1] = 2p - 1 > 0$. The Strong Law of Large Numbers implies that $S_n/n \to 2p - 1$ a.s. and therefore $S_n/n \geq \frac{1}{2}(2p-1)$ for sufficiently large n, so $S_n \to \infty$. By Problem 5 (Exercise 4.1.9), it follows that $P(\alpha < \infty) = 1$ and $P(\beta < \infty) < 1$.

(ii) Define $0 = \beta_0, \beta_1, \ldots$ to be the iterates of β as in Example 4.1.4. Since each $X_n \in \{\pm 1\}$, every new record low value of S_n must be exactly one less than the previous record low. In other words, if $\beta_k < \infty$ then $S_{\beta_k} = -k$. If $\beta_k = \infty$ then $K := \sup\{j \geq 0 : \beta_j < \infty\} < k$. In that case we have $S_{\beta_K} = -K < S_n$ for all $n < \beta_K$ (since the value at time β_K is a record low) and $S_n \geq -K$ for all $n \geq \beta_K$ (since $\beta_{K+1} = \infty$), meaning that $\inf_n S_n = -K > -k$. Hence $\inf_n S_n \leq -k$ if and only if $\beta_k < \infty$, which happens with probability $P(\beta_k < \infty) = P(\beta < \infty)^k$.

(iii) Wald's equation says that $E[S_{\alpha\wedge n}] = E[X_1]E[\alpha \wedge n]$ for each n. Since $\alpha < \infty$ a.s. from part (i), $\alpha \wedge n \nearrow \alpha$ a.s. as $n \to \infty$. By monotone convergence, the right side of the Wald equation converges to $E[X_1]E[\alpha]$ as $n \to \infty$. For the left side we use dominated convergence:

if $Y = \inf_m S_m$ then $Y \leq S_{\alpha\wedge n} \leq 1$. By part (ii),

$$E[|Y|] = \sum_{k=1}^{\infty} P(|Y| \geq k) = \sum_{k=1}^{\infty} P(Y \leq -k) = \sum_{k=1}^{\infty} P(\beta < \infty)^k < \infty$$

where the last inequality is because $P(\beta < \infty) < 1$ by part (i). Since the almost sure pointwise limit of $S_{\alpha\wedge n}$ is S_α (again because α is a.s. finite), dominated convergence gives $E[S_{\alpha\wedge n}] \to E[S_\alpha]$ as $n \to \infty$. Finally we observe that $S_\alpha = 1$ whenever $\alpha < \infty$. Therefore, $E[S_\alpha] = E[X_1]E[\alpha] \implies 1 = (2p-1)E[\alpha] \implies E[\alpha] = 1/(2p-1)$.