Math 6710, Fall 2016
Homework 2

1. Suppose that F is the distribution function of a random variable X. Show that the set

$$\mathcal{D} = \{x \in \mathbf{R} : F \text{ is discontinuous at } x\}$$

is countable (and thus has Lebesgue measure zero).

2. Recall that a "continuous distribution" on $\mathbf{R}$ is a probability measure μ satisfying $\mu(\{x\}) = 0$ for all $x \in \mathbf{R}$. Prove that μ is a continuous distribution if and only if the associated distribution function $F(x) = \mu((-\infty, x])$ is a continuous function.

3. Suppose that F is the distribution function of a random variable X. If F is a continuous function, show that the random variable $Y = F(X)$ has a uniform distribution on $[0, 1]$, that is, $P(Y \leq y) = y$ for all $y \in [0, 1]$.

4. Let F be a distribution function on $\mathbf{R}$ and let μ be the associated probability measure. If μ is absolutely continuous with respect to the Lebesgue measure m, the Radon-Nikodym Theorem provides a density function f that satisfies

$$\int_E f\, dm = \mu(E) \quad \text{for all } E \in \mathcal{B}. \tag{1}$$

In particular, if $E = (-\infty, x]$, one has

$$\int_{-\infty}^{x} f(t)dt = \mu((-\infty, x]) = F(x). \tag{2}$$

Prove that any measurable function f satisfying condition (2) for all $x \in \mathbf{R}$ also satisfies the stronger condition (1). Hint: Consider the collection of all sets E for which (1) holds.

Note: The Fundamental Theorem of Calculus implies that the density function f (when it exists) and the distribution function F are related by $F' = f$ at almost every $x \in \mathbf{R}$, and at every $x \in \mathbf{R}$ when f is continuous.

5. Suppose that X, Y, Z are random variables with $X \overset{\text{dist}}{=} Y$. Can we conclude that $XZ \overset{\text{dist}}{=} YZ$? Prove or give a counterexample.

6. Suppose that X is a map from $(\Omega, \mathcal{F})$ to $(S, \mathcal{G})$. Show that if $\mathcal{A} \subseteq \mathcal{G}$ generates $\mathcal{G}$, then $X^{-1}(\mathcal{A}) = \{X^{-1}(A) : A \in \mathcal{A}\}$ generates $\sigma(X)$.