Grading: 1, 3, 4, 6 (each 5 pts) and 2, 5 (10 pts).

Problem 1: Denote $S_0 = \{w : X_n(w) \to X\}$ and $S_1 = \{w : f(X_n(w)) \to f(X)\}$. Since f is continuous, then on S_0 we have $f(X_n(w)) \to f(X(w))$, so $S_0 \subseteq S_1$ and because $P(S_0) = 1$ we hence get $P(S_1) = 1$, as desired. *(Note: $f(X_i)$ are random variables by Thm 1.3.2 of the book.)*

Problem 2: By Theorem 1.3.2 (with $\mathcal{F} = \sigma(X)$), because f is measurable and X is a random variable, we get that $Y = f(X)$ should also be measurable w.r.t. $\sigma(X)$. For the opposite direction we need to show that if Y is measurable w.r.t. $\sigma(X)$, then the function $f = \lim_{n\to\infty} f_n$ given in the exercise is well-defined and satisfies $Y = f(X)$. First, the set $\{\omega \in \Omega : m2^{-n} \leq Y(\omega) < (m+1)2^{-n}\} = Y^{-1}([m2^{-n}, (m+1)2^{-n})) \in \sigma(X) = \{X^{-1}(B) : B \in \mathcal{B}\}$, so we can write $\{\omega \in \Omega : m2^{-n} \leq Y(\omega) < (m+1)2^{-n}\} = X^{-1}(B_{m,n})$ for some $B_{m,n} \in \mathcal{B}$.

We would like to say that the sets $B_{m,n}$ are disjoint for fixed n, set $f_n = \sum_{m\in\mathbb{Z}} m2^{-n}\mathbf{1}_{B_{m,n}}$, and then claim that the sequence f_n is increasing ($f_n(x)$ is a nondecreasing sequence for each $x \in \mathbb{R}$). This works fine when the range $X(\Omega)$ is all of $\mathbb{R}$, but there are complications when $X(\Omega) \subsetneq \mathbb{R}$. In fact, $X(\Omega)$ need not even be a measurable set. So we argue carefully as follows.

Start with $n = 0$. If the sets $B_{m,0}$ are not already disjoint, choose an ordering $(m_1, m_2, \ldots)$ of $\mathbb{Z}$ such as $(0, 1, -1, 2, -2, \ldots)$ and define $B'_{m_j,0} = B_{m_j,0} \setminus \cup_{i=1}^{j-1} B_{m_i,0}$. The sets $B'_{m,0}$ are in $\mathcal{B}$ and disjoint. In addition, each $X^{-1}(B'_{m_j,0}) = X^{-1}(B_{m_j,0})$, by the following reasoning. $B_{m_j,0}$ is the disjoint union of $B'_{m_j,0}$ with the set $B_{m_j,0} \cap (\cup_{i=1}^{j-1} B_{m_i,0})$, which we denote by C_j. If $X(\omega) \in C_j$ for some $\omega \in \Omega$, then $X(\omega)$ is in both $B_{m_j,0}$ and $B_{m_i,0}$ for some $i < j$, so $Y(\omega)$ is simultaneously in two disjoint intervals, which is not possible. Hence $X^{-1}(C_j) = \varnothing$ and $X^{-1}(B'_{m_j,0}) = X^{-1}(B_{m_j,0})$. Since we could replace with $B'_{m,0}$ if necessary, we may assume that the sets $B_{m,0}$ are disjoint.

Now we inductively (on n) modify the sets $B_{m,n}$ for $n \geq 1$. Since $X^{-1}(B_{2k,n} \cup B_{2k+1,n}) = X^{-1}(B_{k,n-1})$, the sets $B_{2k,n} \cap X(\Omega)$ and $B_{2k+1,n} \cap X(\Omega)$ are disjoint with union $B_{k,n-1} \cap X(\Omega)$. These sets may be non-measurable, so we instead replace $B_{2k,n}$ with $B'_{2k,n} = B_{2k,n} \cap B_{k,n-1}$ and replace $B_{2k+1,n}$ with $B'_{2k+1,n} = B_{k,n-1} \setminus B'_{2k,n}$. It can be shown that $X^{-1}(B'_{2k,n}) = X^{-1}(B_{2k,n})$ and likewise for $B'_{2k+1,n}$. Do this for all $k \in \mathbb{Z}$ and then suppress the primes in the notation before moving on to $n+1$. In the end, the sets $B_{m,n}$ are disjoint for fixed n, and each $B_{k,n-1} = B_{2k,n} \sqcup B_{2k+1,n}$.

Set $f_n = \sum_{m\in\mathbb{Z}} m2^{-n}\mathbf{1}_{B_{m,n}}$. Note that $f_n = \sum_{k\in\mathbb{Z}} \left[2k2^{-n}\mathbf{1}_{B_{2k,n}} + (2k+1)2^{-n}\mathbf{1}_{B_{2k+1,n}}\right] \geq \sum_{k\in\mathbb{Z}} 2k2^{-n}\mathbf{1}_{B_{k,n-1}} = f_{n-1}$. In addition, $f_n \leq \sum_{k\in\mathbb{Z}}(2k+1)2^{-n}\mathbf{1}_{B_{k,n-1}} \leq f_{n-1} + 2^{-n}$. It follows that the sequence f_n is increasing and bounded above by $f_0 + 1$, which is finite everywhere, so the function $f = \lim_{n\to\infty} f_n$ is well-defined, measurable, and finite. The construction implies that each $|Y(\omega) - f_n(X(\omega))| < 2^{-n}$, so $f(X(\omega)) = \lim_{n\to\infty} f_n(X(\omega)) = Y(\omega)$. In other words, $Y = f(X)$ as desired.

Problem 3: By the definition of $\|f\|_\infty$, $|f| \leq \|f\|_\infty$ almost surely, i.e. on a set Ω_0 with

$\mu(\Omega_0) = 1$. Since $\mu(\Omega_0^c) = 0$, an integral over a null set is 0 and $\|f\|_\infty$ is a constant, we get $(\|f\|_p)^p = \int_\Omega |X|^p = \int_{\Omega_0} |X|^p \le \int_{\Omega_0} \|f\|_\infty^p = \|f\|_\infty^p$. Hence, $\limsup_{p\to\infty} \|f\|_p \le \|f\|_\infty$ and we are left to show $\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty$. For that, since $\|f\|_\infty = \inf\{m : \mu(\{w \in \Omega : |f(w)| \ge m\} = 0\}$, for any $\epsilon > 0$ there exists finite $\delta > 0$ s.t. $|f| > \|f\|_\infty - \epsilon$ on a set S of measure δ. Hence, $\int_\Omega |f|^p \ge \int_S (\|f\|_\infty - \epsilon)^p \ge \delta(\|f\|_\infty - \epsilon)^p$ so $\|f\|_p \ge \delta^{1/p}(\|f\|_\infty - \epsilon)$. By taking $p \to \infty$ we get $\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty - \epsilon$. Since ϵ was arbitrary it follows that $\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty$, as desired.

Problem 4: Note $\phi(x) = e^x$ is convex, so by Jensen's inequality $E(e^X) \ge \exp(E(X))$, where $RHS = \sum_{m=1}^n p(m) e^{\log y_m} = \sum_{m=1}^n p(m) y_m$ and $LHS = \exp(\sum_{m=1}^n p(m) \log y_m) = \prod_{m=1}^n y_m^{p(m)}$.

Problem 5: a) Because $Z \ge 0$ we have $P'(S) = E(\mathbf{1}_S Z) \ge 0$ for $S \in \mathcal{F}$ and since $E(Z) = 1$ we get $P'(S) = \int_S Z dP \le \int_\Omega Z dP = E(Z) = 1$. So, $P' \le 1$, $P'(\Omega) = 1$ and for a countable disjoint collection $\{S_i\}_{i\in I} \in \mathcal{F}$ we have $P'(\cup_I S_i) = \int_{\cup_I S_i} Z dP = \sum_I \int_{S_i} Z dP = \sum_I P'(S_i)$, which implies that P' is a probability measure. **b)** Similar to constructing the integral, we show the equality for increasingly general cases of X. Note that $E'(X) = E[(X^+ - X^-)Z]$ and it's enough to consider independently X^+ and X^-. The cases $X = \mathbf{1}_S$ for $S \in \mathcal{F}$ and $X = \sum_{i=1}^n c_i \mathbf{1}_{S_i}$ are trivial. Finally for (WLOG) $X \ge 0$, $\exists$ sequence of simple functions $\psi_n \uparrow X$ so by using Monotone Convergence Theorem we get $E'(X) = \lim_{n\to\infty} E'(\psi_n) = \lim_{n\to\infty} E(\psi_n Z) = E(XZ)$, as desired.

Problem 6: a) Since $Z = Y^q/E(Y^q) \ge 0$, $E(Z) = E(Y^q/E(Y^q)) = E(Y^q)/E(Y^q) = 1$ and by using 5b) we get $E'(XY^{1-q}) = E(XY^{1-q}(Y^q/E(Y^q))) = E(XY)/E(Y^q)$, which is equivalent to $E(Y^q)E'(XY^{1-q}) = E(XY)$, as desired. **b)** Since the function x^p for $x \ge 0$ and $p > 1$ is convex, using Jensen's inequality for $W \ge 0$ we get $E'(W)^p \le E'(W^p)$ so $E'(XY^{1-q}) \le E'((XY^{1-q})^p)^{1/p}$. Using 5b), 6a) and that $p + q = pq$ we get that the latter inequality is equivalent to $E(XY) \le E(X^p Y^{p(1-q)} Y^q)^{1/p} E(Y^q)^{1/q} = E(X^p)^{1/p} E(Y^q)^{1/q} = \|X\|_p \|Y\|_q$ which is the Holder's inequality.