Math 6710, Fall 2016
Homework 3

1. Durrett Exercise 1.3.3. *Hint:* Recall that if $f : \mathbf{R} \to \mathbf{R}$ is continuous and x_n is a convergent sequence of real numbers, then

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

2. Durrett Exercise 1.3.9. Make sure to prove both directions of the statement in Exercise 1.3.8; the construction in 1.3.9 should give you the more difficult direction.

3. Durrett Exercise 1.5.2. *Hint:* Suppose $P(|f(\omega)| \geq c) \geq \varepsilon$. Get a lower bound on $\|f\|_p$ and send p to infinity.

4. Durrett Exercise 1.6.12.

5. Let $Z \geq 0$ be a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$ with $E[Z] = 1$. Define a function $P' : \mathcal{F} \to \mathbf{R}$ by

$$P'(A) = E[\mathbf{1}_A Z] = \int_A Z \, dP.$$

(a) Show that P' is a probability measure.

(b) Let E' denote expectation with respect to the probability measure P'. That is, for measurable $X : \Omega \to \mathbf{R}$, $E'[X] = \int X \, dP'$. Show that $E'[X] = E[XZ]$ whenever either side is defined. *Hint:* Follow the steps used to prove Theorem 5.8 in the lecture notes.

6. This problem gives an alternate proof of Hölder's inequality. Let $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Consider random variables X and Y; without loss of generality we may assume that $X, Y \geq 0$ (since both sides of the inequality involve absolute values), and we can also assume that $\|X\|_p, \|Y\|_q > 0$ (else the inequality reduces to $0 \leq 0$).

(a) Set $Z = Y^q / E[Y^q]$. Check that $E[Z] = 1$. Use this Z to define the expectation E' as in Problem 5, and verify that

$$E[XY] = E[Y^q] E'[XY^{1-q}].$$

(b) Explain why $E'[W] \leq E'[W^p]^{1/p}$ for any random variable $W \geq 0$. Use this fact for $W = XY^{1-q}$ along with part (a) to prove Hölder's inequality.