Math 6710, Fall 2016
Homework 4

1. For real $p > 0$, the pth moment of a random variable X is $E[X^p]$.

(a) Fix $r > 0$ and let X have distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 1, \\ 1 - x^{-r} & \text{if } x \geq 1. \end{cases}$$

For which values of $p > 0$ does X have finite pth moment? Compute the pth moment when it is finite.

(b) Find a distribution function $F(x)$ with $F(0) = 0$ such that if X has distribution F, then the pth moment of X is infinite for every real $p > 0$.

2. Consider the definitions of independence for events, random variables, and σ-algebras given in the lecture notes (and in Section 2.1 of Durrett).

(a) Show that if the events A and B are independent, then so are A and B^C. Use this to argue that A and B are independent if and only if the indicator variables $\mathbf{1}_A$ and $\mathbf{1}_B$ are independent.

(b) Show that if the random variables X and Y are independent, then the σ-algebras $\sigma(X)$ and $\sigma(Y)$ are independent. Conversely, suppose that X is measurable as a function from $(\Omega, \mathcal{F}_X)$ to $(\mathbf{R}, \mathcal{B})$ (which Durrett abbreviates as "$X \in \mathcal{F}_X$") and also $Y \in \mathcal{F}_Y$. Show that if $\mathcal{F}_X$ and $\mathcal{F}_Y$ are independent, then X and Y are independent.

3. Durrett Exercise 2.1.11.

4. Suppose that X is a random variable with distribution μ. Let $f, g : \mathbf{R} \to \mathbf{R}$ be nondecreasing functions in $L^2(\mu)$, that is, $\int |f|^2 \, d\mu < \infty$ and likewise for g. Prove that $E[f(X)g(X)] \geq E[f(X)]E[g(X)]$. Where in your proof did you use that $f, g \in L^2(\mu)$? *Hint:* Let Y be an independent copy of X and consider $[f(X) - f(Y)][g(X) - g(Y)]$.

5. Let $X_1, X_2, \ldots$ be an iid (independent and identically distributed) sequence of random variables. Assume that each $X_i > 0$ almost surely, and that both $E[X_i]$ and $E[1/X_i]$ are finite. Set $S_n = X_1 + \cdots + X_n$. Show that $E[S_m/S_n] = m/n$ when $m \leq n$ and that $E[S_m/S_n] = 1 + (m - n)E[X_1]E[1/S_n]$ when $m > n$.