Grading: 1, 2, 3, 4, 5 (each 8 pts).

Problem 1: a) $P(|X_n - X| > \epsilon) = P(|a_n - a| > \epsilon) = \mathbf{1}\{|a_n - a| > \epsilon\}$. $X_n \to_p X$ if the left side goes to zero as $n \to \infty$ for each fixed ϵ, and $a_n \to a$ if the right side goes to zero as $n \to \infty$ (i.e. is eventually all zeros) for each fixed ϵ. **b)** For the first statement, fix $\epsilon > 0$. By the triangle inequality and a union bound, $P(|Z_n + Z_n'| > \epsilon) \leq P(|Z_n| > \epsilon/2) + P(|Z_n'| > \epsilon/2)$ and both those terms tend to zero as $n \to \infty$. For the second statement, take $Z_n = X_n - X$ and $Z_n' = Y_n - Y$. Then $Z_n \to_p 0$ and $Z_n' \to_p 0$, so $Z_n + Z_n' \to_p 0$ which is equivalent to the desired statement $X_n + Y_n \to_p X + Y$.

Problem 2: Fix $\epsilon, \delta > 0$ and choose $k = k(\delta)$ sufficiently large that $P(|X| > k) < \delta$ and $P(|Y| > k) < \delta$. By additivity, $P(|X_nY_n - XY| > \epsilon) \leq P(|X_nY_n - XY| > \epsilon, |X| \leq k, |Y| \leq k) + P(|X| > k) + P(|Y| > k)$. Next, for $|X| \leq k$ and $|Y| \leq k$, note that by the triangle inequality $|X_nY_n - XY| = |(X_n - X)(Y_n - Y) + X(Y_n - Y) + Y(X_n - X)| \leq |X_n - X||Y_n - Y| + k|X_n - X| + k|Y_n - Y|$. If both $|X_n - X|, |Y_n - Y| \leq \epsilon/(3k)$ then this sum is at most $\epsilon^2/(9k^2) + \epsilon/3 + \epsilon/3$ which is at most ϵ as long as $\epsilon/(3k^2) \leq 1$ (which we may assume by increasing k if necessary). Therefore $P(|X_nY_n - XY| > \epsilon, |X| \leq k, |Y| \leq k) \leq P(|X_n - X| > \epsilon/(3k)) + P(|Y_n - Y| > \epsilon/(3k))$ which tends to zero as $n \to \infty$ for fixed ϵ, k. We conclude that $\lim_{n\to\infty} P(|X_nY_n - XY| > \epsilon) \leq 2\delta$. Since δ was arbitrary this limit must be zero, so $X_nY_n \to_p XY$.

Problem 3: For any $0 < \epsilon < 1$, $0 \leq d(X, Y) = E(\min\{1, |X - Y|\}) = E(\mathbf{1}_{|X-Y|>1} + |X - Y|\mathbf{1}_{|X-Y|\leq 1}) = P(|X - Y| > 1) + E(|X - Y|\mathbf{1}_{|X-Y|\leq 1}) = P(|X - Y| > 1) + E(|X - Y|\mathbf{1}_{\epsilon<|X-Y|\leq 1}) + E(|X - Y|\mathbf{1}_{|X-Y|\leq\epsilon})$. Next, suppose $X_n \to_p X$. For any $0 < \epsilon < 1$, if n is sufficiently large then $P(|X - X_n| > \epsilon) \leq \epsilon$. Thus, $d(X, X_n) = P(|X - X_n| > 1) + E(|X - X_n|\mathbf{1}_{\epsilon<|X-X_n|\leq 1}) + E(|X - X_n|\mathbf{1}_{|X-X_n|\leq\epsilon}) \leq P(|X - X_n| > 1) + 1 \cdot P(\epsilon < |X - X_n| \leq 1) + \epsilon \cdot P(|X - X_n| \leq \epsilon) \leq P(|X - X_n| > \epsilon) + \epsilon \cdot 1 \leq 2\epsilon$. It follows that $d(X, X_n) \to 0$. To prove the reverse, suppose that $d(X, X_n) \to 0$ and fix $\epsilon > 0$. For every $\epsilon_1 > 0$ there exists n_0 s.t. $n \geq n_0$ implies $d(X, X_n) < \epsilon\epsilon_1$. So, $\epsilon\epsilon_1 > d(X, X_n) = P(|X - X_n| > 1) + E(|X - X_n|\mathbf{1}_{\epsilon<|X-X_n|\leq 1}) + E(|X - X_n|\mathbf{1}_{|X-X_n|\leq\epsilon}) \geq P(|X - X_n| > 1) + \epsilon P(|X - X_n| > \epsilon) \geq \epsilon(P(|X - X_n| > 1) + P(|X - X_n| > \epsilon)) = \epsilon P(|X - X_n| > \epsilon)$, that is $P(|X - X_n| > \epsilon) < \epsilon_1$ for $n \geq n_0$. Now since ϵ_1 was arbitrary we get $P(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$, as desired.

Problem 4: Since the X_i are uncorrelated, $E[(S_n/n - \nu_n)^2] = \mathrm{Var}(S_n/n) = (1/n^2)[\mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)] = (1/n)[\mathrm{Var}(X_1)/n + \cdots + \mathrm{Var}(X_n)/n] \leq (1/n)[\mathrm{Var}(X_1)/1 + \mathrm{Var}(X_2)/2 + \cdots + \mathrm{Var}(X_n)/n]$. If $a_n = \mathrm{Var}(X_n)/n$ then $a_n \to 0$ and so the averages $(1/n)(a_1 + \cdots + a_n)$ also tend to 0. (Proof: For any $\epsilon > 0$ there is n_0 such that $|a_n| < \epsilon$ for all $n > n_0$. Write $(1/n)(a_1 + \cdots + a_n) = (1/n)(a_1 + \cdots + a_{n_0}) + (1/n)(a_{n_0+1} + \cdots + a_n)$. The first term tends to 0 as $n \to \infty$ and the second term is less than ϵ in absolute value, hence

$\limsup_{n\to\infty}(1/n)|a_1 + \cdots + a_n| \le \epsilon$. Since ϵ was arbitrary the limit must be zero.) It follows that $E[(S_n/n - \nu_n)^2] \to 0$, hence $S_n/n - \nu_n \to 0$ in L^2 and thus also in probability.

Problem 5: a) Because it is integer valued, then $X = \sum_{n=1}^{\infty} \mathbf{1}_{X\ge n}$, so $E[X] = E[\sum_{n=1}^{\infty} \mathbf{1}_{X\ge n}] = \sum_{n=1}^{\infty} P(X \ge n)$.

b) $E[X^2] = \sum_{n=1}^{\infty} n^2 P(X = n) = \sum_{n=1}^{\infty} \sum_{m=1}^{n^2} P(X = n) = \sum_{m=1}^{\infty} \sum_{n:n^2\ge m} P(X = n) = \sum_{m=1}^{\infty} P(X \ge \sqrt{m}) = \sum_{k=1}^{\infty} \sum_{m=(k-1)^2+1}^{k^2} P(X \ge k) = \sum_{k=1}^{\infty}[k^2 - (k-1)^2]P(X \ge k) = \sum_{k=1}^{\infty}(2k-1)P(X \ge k)$. Alternatively, using the Lemma 2.2.8 of the textbook and the fact that X is integer-valued we have $E[X^2] = \int_0^{\infty} 2xP(X > x)dx = \sum_{n=1}^{\infty} \int_{n-1}^{n} 2xP(X > x)dx = \sum_{n=1}^{\infty} P(X \ge n)[n^2 - (n-1)^2]$, which is the same as above.