Math 6710, Fall 2016
Homework 6

1. Let $X_1, X_2, \dots$ be iid with finite variance $\mathrm{Var}(X_1) = \sigma^2$. The sample mean of the first n terms is $\overline{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$ and the sample variance is $\overline{V}_n = \frac{1}{n-1}\sum_{k=1}^n (X_k - \overline{X}_n)^2$. Show that $E[\overline{V}_n] = \sigma^2$ for all $n \geq 2$ and that $\overline{V}_n \to \sigma^2$ in probability.

2. Let $X_1, X_2, \dots$ be iid with $E[X_1] = \mu$ and $\|X_1\|_p < \infty$ for fixed $1 \leq p \leq 2$. Set $S_n = X_1 + \cdots + X_n$. The purpose of this problem is to prove that $\frac{1}{n}S_n \to \mu$ in L^p. Note that we may assume without loss of generality that $\mu = 0$.

You may find the following facts useful: $\|X\|_q \leq \|X\|_r$ for $q \leq r$ (which we proved using Jensen's inequality) and $\|X + Y\|_q \leq \|X\|_q + \|Y\|_q$ (which can be proved using Hölder's inequality, though we skipped this).

(a) Define W_i and Z_i as in the proof of Theorem 7.3 in the notes. Argue that $\|Z_1\|_p \leq \|X_1 \mathbf{1}\{|X_1| > C\}\|_p + \|X_1 \mathbf{1}\{|X_1| > C\}\|_1$. Explain why for any $\delta > 0$ there exists C that makes $\|Z_1\|_p < \delta$.

(b) Show that $\|S_n\|_p \leq \|W_1 + \cdots + W_n\|_2 + n\|Z_1\|_p$, and use this to prove that $\frac{1}{n}S_n \to 0$ in L^p as $n \to \infty$.

3. Durrett Exercise 2.2.3.

4. Durrett Exercise 2.2.4.

5. Durrett Exercise 2.2.8. In the print version of the textbook, the mysterious (5.5) refers to Theorem 2.2.6, which is Theorem 7.4 in the lecture notes.